12.510 Introduction to Seismology

Lecture 5

Feb. 22, 2005

1 Introduction

At previous lectures, we derived the equation of motion

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}(\mathbf{x}, t)) - \mu\nabla \times (\nabla \times \mathbf{u}(\mathbf{x}, t)) = \rho\frac{\partial^{\mathbf{2}}\mathbf{u}(\mathbf{x}, t)}{\partial t^{\mathbf{2}}} \tag{1}$$

This equation of motion can be expressed by a *scalar potential*, $\phi(\mathbf{x}, t)$, and a *vector potential*, $\Psi(\mathbf{x}, t)$ which are *Helmholtz potentials*, based on the relations $\mathbf{u}(\mathbf{x}, t) = \nabla\phi(\mathbf{x}, t)+\nabla\times\mathbf{\Psi}(\mathbf{x}, t)$ and $\nabla \cdot \mathbf{\Psi}(\mathbf{x}, t) = \mathbf{0}$. The two potentials give us two wave equations. The scalar potential satisfies

$$\nabla^2\phi(\mathbf{x}, t) = \frac{\mathbf{1}}{\alpha^{\mathbf{2}}}\frac{\partial^{\mathbf{2}}\phi(\mathbf{x}, t)}{\partial t^{\mathbf{2}}} \tag{2}$$

with the velocity $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$. Similarly, the vector potential satisfies

$$\nabla^2\Psi(\mathbf{x}, t) = \frac{\mathbf{1}}{\beta^{\mathbf{2}}}\frac{\partial^{\mathbf{2}}\mathbf{\Psi}(\mathbf{x}, t)}{\partial t^{\mathbf{2}}} \tag{3}$$

with velocity $\beta = (\mu/\rho)^{1/2}$. These two equations can be solved by the following three ways: *d'Alembert's solution, separation of variables* and *Fourier Transform*. Although they are different methods to get solution, they give the same solution of equation of motion. This lecture explains not only the three ways to get plane wave solutions, but also the concepts of *dispersion relation* and *slowness*. In addition, this lecture introduces the nomenclature of body waves in Earth's interior.

2 d'Alembert's Solution

We begin with the one-dimensional function,

$$f(\pm x \mp ct). \tag{4}$$

The phase of this function is x - ct, so if we only consider constant phase, we can get phase velocity, $c = x/t$. For convenience, if we consider slightly different form, x/c -t, multiplied by angular frequency, ω,and using the relation of $k = \omega/c$, we can get more convenient form, (kx - ωt). In three-dimensional case, the phase becomes $k_x x + k_y y + k_z z - \omega t = \mathbf{k} \cdot \mathbf{x} - \omega t$ for the following complex form wave functions:

$$\phi = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad\qquad \Psi = \mathbf{B} \times \hat{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{5}$$

Equation (5) represents the solutions of wave equation (2) and (3).

3 Separation of variables

Separation of variables is a method of solving ordinary and partial differential equations. For a partial differential equation, we can use separation of variables to make a substitution of the form

$$\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \tag{6}$$

to equation (2), and divide it by $XYZT$, then we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0 \tag{7}$$

To satisfy this equation (7), each term has to be constant ${k_x}^2, {k_y}^2, {k_z}^2$ and ω^2/c^2, respectively. Each term makes four ODEs:

$$\begin{aligned}
\frac{d^2X}{dx^2} + {k_x}^2 X = 0 &\longrightarrow e^{\pm ik_x X} \\
\frac{d^2Y}{dy^2} + {k_y}^2 Y = 0 &\longrightarrow e^{\pm ik_y Y} \\
\frac{d^2Z}{dz^2} + {k_z}^2 Z = 0 &\longrightarrow e^{\pm ik_z Z} \\
\frac{d^2T}{dt^2} + \omega^2 T = 0 &\longrightarrow e^{\pm i\omega T}
\end{aligned}$$

From the equation (6), these equations produce desirable solution form, $\phi = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Now we can get the full solution to make use of superposition of plane waves.

$$\phi(\mathbf{k}, \mathbf{x}) = \sum_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{8}$$

To make each term in equation (7) constant provides physically important phenomenon, what is called *dispersion*.

$${k_x}^2 + {k_y}^2 + {k_z}^2 - \frac{\omega^2}{c^2} = 0 \tag{9}$$

The dispersion relation means that waves with different frequencies travel with different velocities. Furthermore, if we know three of four parameters, one of them can be fixed from the relation (9). If we consider one-dimensional solution form as $\phi = e^{i(kx_x - \omega t)}$, then $d\phi/dx = ik_x\phi$, $d^2\phi/dx^2 = -{k_x}^2\phi$. From equation (2),

$$\ddot{\phi} = \alpha^2\nabla^2\phi = -\alpha^2{k_x}^2\phi \tag{10}$$

$$\ddot{\phi} = \frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi \tag{11}$$

Thus, we can get the dispersion relation,

$${k_x}^2 = \left(\frac{\omega}{\alpha}\right)^2 \tag{12}$$

We can also get the *Helmholtz equation*.

$$\nabla^2\phi + k^2\phi = 0 \tag{13}$$

We will get back to the dispersion relation later.

4 Fourier Transform

Fourier transform lets us understand the relation between space–time $(\mathbf{x}, t)$ domain and wave number–frequency $(\mathbf{k}, \omega)$ domain. The followings are Fourier transform and inverse Fourier transform in term of time–frequency and space-wave number, respectively.

$$\Phi(\mathbf{x},\omega) = \int_{-\infty}^{\infty}\phi(\mathbf{x},t)e^{i\omega t}dt \longleftrightarrow \phi(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{\Phi}(\mathbf{x},\omega)e^{-i\omega t}d\omega \tag{14}$$

$$\Phi(\mathbf{k},t) = \int_{\mathbf{V}}\phi(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{r}}d^3r \longleftrightarrow \phi(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int_{\mathbf{k}}\mathbf{\Phi}(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{x}}dk_xdk_ydk_z \tag{15}$$

We combine the two relation, i.e. double Fourier transform

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\iint_{-\infty}^{\infty}\mathbf{\Phi}(k_x,k_y,\omega,z)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}dk_xdk_yd\omega \tag{16}$$

This is the solution that satisfies the wave equation (2). It is also shown that this solution is similar to the solution (8) obtained by separation of variables. The integrand, $\Phi(k_x, k_y, \omega, z)$, can be considered as amplitude or weight. There are two points to consider the equation (16) in detail . First, we do not need to put z component of wave number in the integration because of the dispersion relation (9). As mentioned before, one parameter can be fixed by dispersion relation if we know three of four parameters, i.e. $k_z = f(k_x, k_y, \omega)$. Second, it is difficult in practice to integrate from $-\infty$ to ∞ in terms of wave number and from

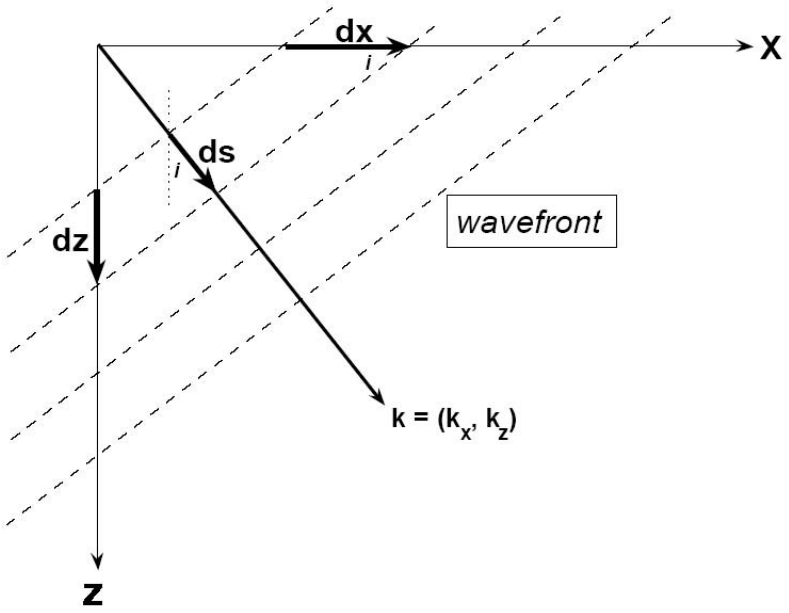


Figure 1: This figure describe the ray and wavefront. The arrow is used for a ray and dash–line for wavefront. The wave number $\mathbf{k}$ indicates direction of the ray. The angle i is take–off angle or angle of incidence. ds, dx and dz are distance along the ray, horizon and vertical, respectively. We can get the slowness vector from this figure.

$-\pi$ to π in frequency. However, it can be used to produce synthetic seismogram by limit: integration over directions $k_0 \pm dk$ and frequency $\omega_0 \pm d\omega$.

$$\phi(\mathbf{x}, t) = \frac{\mathbf{1}}{(\mathbf{2}\pi)^{\mathbf{3}}} \int_{\omega_0 - d\omega}^{\omega_0 + d\omega} \iint_{k_0 - dk}^{k_0 + dk} \mathbf{\Phi}(k_x, k_y, \omega, z) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} dk_x dk_y d\omega \tag{17}$$

The full solution can be obtained by the superposition of plane wave like equation (8), and we find the displacement by $\mathbf{u}(\mathbf{x}, t) = \nabla\phi(\mathbf{x}, t)$.

5 Slowness

Now we define the slowness vector, so we can easily understand what the dispersion relation means. We define the ray speed, $c = ds/dt$, horizontal wave speed, $c_x = dx/dt$, and vertical wave speed, $c_z = dz/dt$. From [Figure 1], we can relate the angle of incidence with horizontal and vertical wave speed as below.

$$sin(i) = \frac{ds}{dx} = c\frac{dt}{dx} = \frac{c}{c_x} \equiv cp \tag{18}$$

$$cos(i) = \frac{ds}{dz} = c\frac{dt}{dz} = \frac{c}{c_z} \equiv c\eta \tag{19}$$

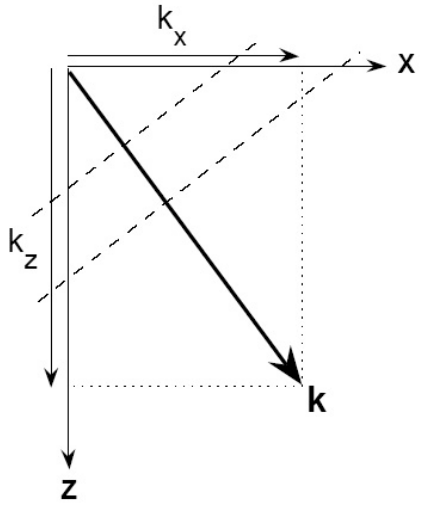


Figure 2: This figure describe the ray in terms of wave number. The arrow is used for a ray and dash–line for wavefront. The wave number **k** which magnitude is ω/c, indicates direction of the ray. Here $k_x = \omega p$ and $k_z = \omega\eta$

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Here p is *horizontal slowness* and η is *vertical slowness*.

$$p \equiv \frac{1}{c_x} = \frac{sin(i)}{c} \qquad\qquad \eta \equiv \frac{1}{c_z} = \frac{cos(i)}{c} \tag{20}$$

The *slowness vector* is composed of horizontal and vertical slowness, $\mathbf{s} = (p, \eta)$. Let us examine some properties of slowness vector. From equation (20),

$$s = \sqrt{p^2 + \eta^2} = \frac{1}{c} \qquad\qquad p^2 + \eta^2 = \frac{1}{c^2} \tag{21}$$

However, the addition of squares of horizontal wave speed and vertical wave speed does not equal to squares of wave speed, ${c_x}^2 + {c_z}^2 \neq c^2$. In addition, we will examine critical phenomenon in reflection and refraction with the relation (21), $\eta = \sqrt{1/c^2 - p^2}$. In terms of wave number, each component of wave number can be represented by horizontal and vertical slowness.

$$k_x = \frac{\omega}{c_x} = \omega p \qquad\qquad k_z = \frac{\omega}{c_z} = \omega\eta \tag{22}$$

Thus, wave number is related to slowness vector.

$$\mathbf{k} = (k_x, k_y) = (\omega p, \omega\eta) = \omega(p, \eta) = \omega\mathbf{s} \tag{23}$$

As shown in [Figure 2], if we know k_x which is related to horizontal slowness and ω/c, then k_z is fixed. This situation is based on dispersion relation.

6 Nomenclature of body waves in Earth's interior

At this stage, it is useful to introduce the jargon used to describe the different types of body wave propagation in Earth's interior. There are a few simple basic rules, but there are also some inconsistencies.
Capital letters are used to denote body wave propagation (transmission) through a medium[1]. On the other hand, lower case letters are used to indicate either reflections or upward propagation of body waves before they are reflected at Earth's surface. Note that this is always used in combination of a transmitted wave. The followings are summary of nomenclature of body wave.

P	a P-wave in the mantle
S	an S-wave in the mantle
K	a P-wave through the outer core
I	a P-wave through the inner core
J	an S-wave through the inner core
c	a reflection from the mantle – outer core boundary
i	a reflection from the outer core – inner core boundary
p	a up-going P-wave from the earthquake focus to the surface of the earth
s	a up-going S-wave from the earthquake focus to the surface of the earth
LR	a Rayleigh wave
LQ	a Love wave

The [Figure 3] and [Figure 4][2] describe the ray paths in terms of various phases. Note that the phases are in combination of a wave described above.

Notes: Kang Hyeun Ji (Feb. 22, 2005)

[1] The phase J has no definitive observations of this seismic phase, although recent research has produced compelling evidence for its existence

[2] Adapted from B.L.N. Kennett (2001), *The Seismic Wavefield Volume 1: Introduction and Theoretical Development*, pp. 78 – 80, Cambridge University Press

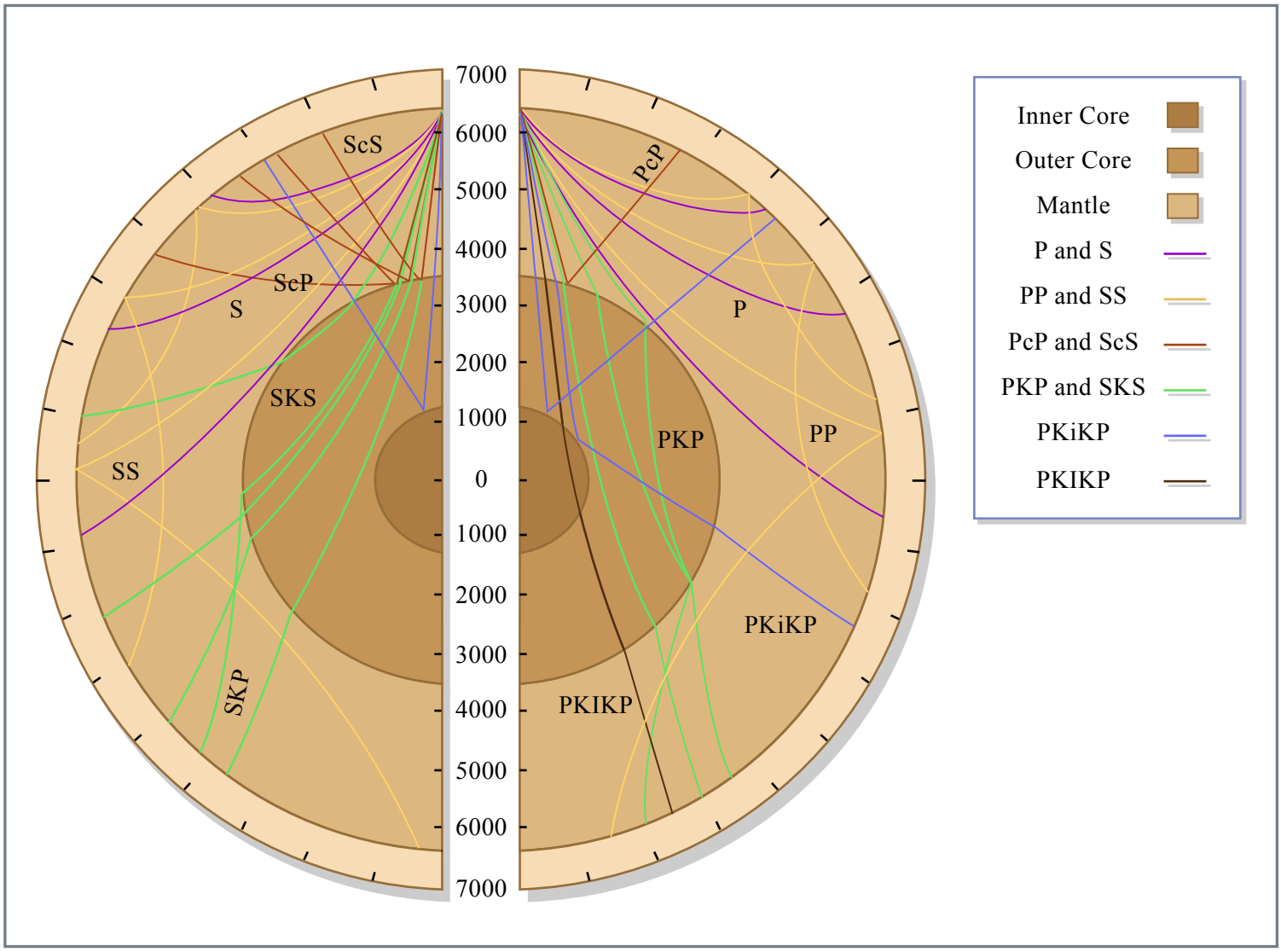


Figure by MIT OCW.

Figure 3: The main seismic phase in the Earth.

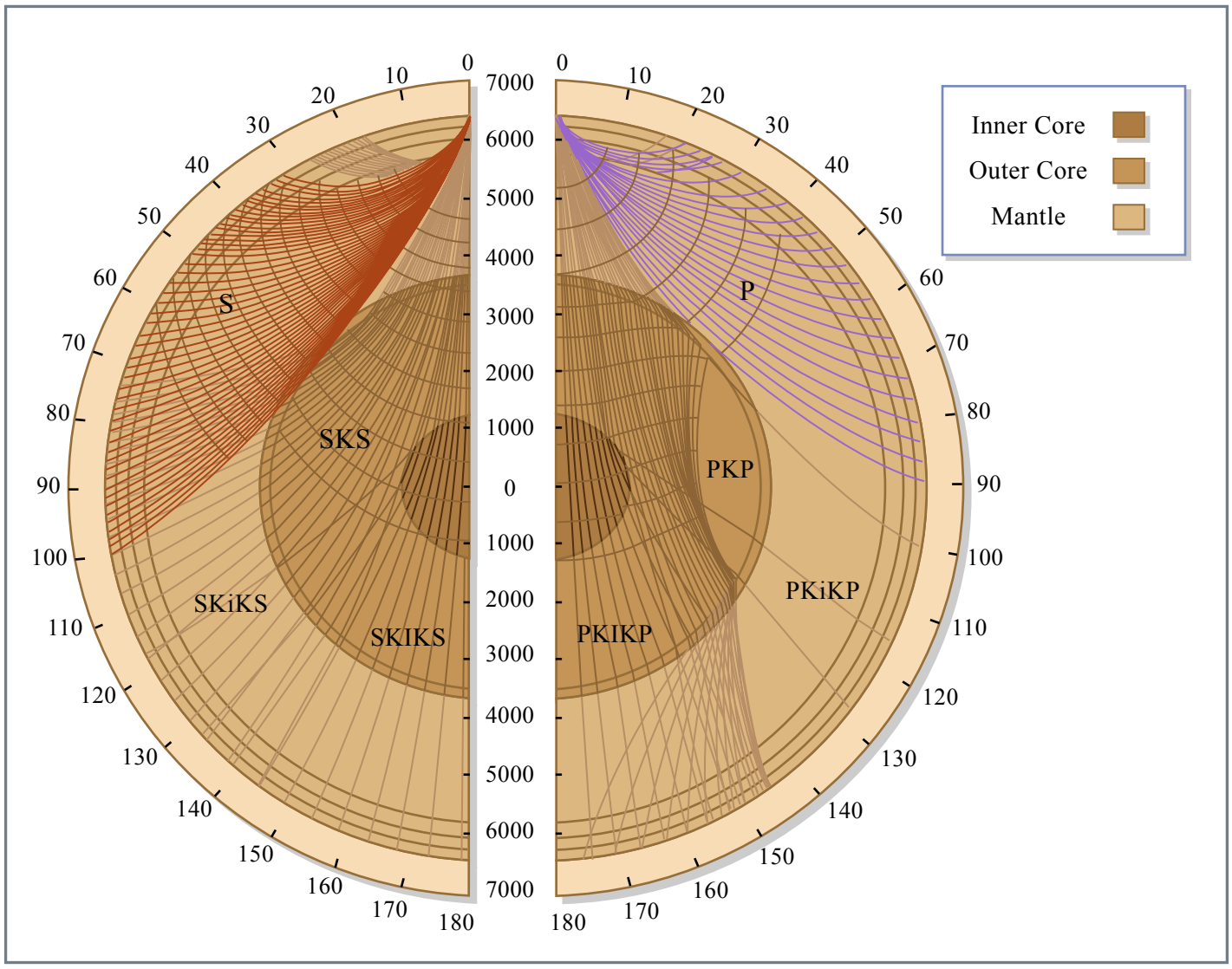


Figure by MIT OCW.

Figure 4: Ray paths for major P and S phases for the AKI35 model of seismic wavespeeds.