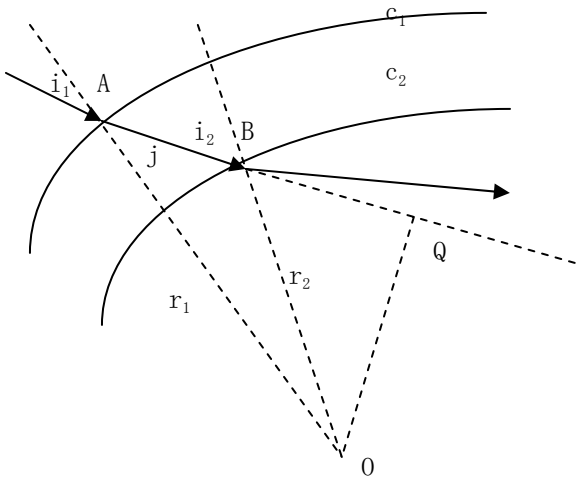
TODAY'S LECTURE

1. Snell's law in spherical media
2. Ray equation
3. Radius of curvature
4. Amplitude → Geometrical spreading
5. $\tau - p$

SNELL'S LAW IN THE SPHERICAL MEDIA



At each interface

$$\frac{\sin i_1}{c_1} = \frac{\sin j}{c_2}$$

$$\sin j = \frac{OQ}{OA} \quad \sin i_2 = \frac{OQ}{OB}$$

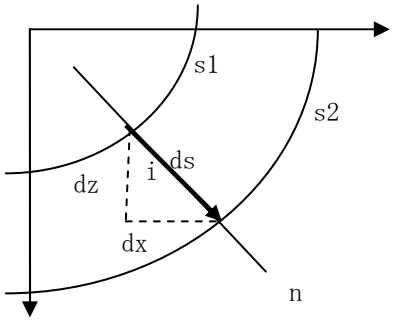
$$\sin j = \frac{OB}{OA}\sin i_2 = \frac{r_2}{r_1}\sin i_2$$

$$\boxed{\frac{r_1 \sin i_1}{c_1} = \frac{r_2 \sin i_2}{c_2} \equiv p}$$

"flat earth" → $p = \dfrac{\sin i}{c}$ "spherical earth" → $p = \dfrac{r \sin i}{c}$

At critical angle, $p = \dfrac{r_p}{c(r_p)}$ we can get depth of layer.

RAY EQUATION



Directional cosine (3D and 2D)

$$\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 = 1 \qquad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$$

Direction of ray ($\hat{n}$)

$$\hat{n} = (n_x, 0, n_z) \qquad n_x = \frac{dx}{ds} \qquad n_z = \frac{dz}{ds}$$

Using Eikonal equation $\nabla T = \frac{1}{c}\hat{n}$,

Generalized Snell's law (Ray Equation)

$$\boxed{\frac{d}{dx_i}\left(\frac{1}{c(x)}\right) = \frac{d}{ds}\left(\frac{1}{c(x)}\frac{dx_i}{ds}\right)}$$

This equation means that the change of wavespeed is related to change of ray geometry. If there is no change in x direction, the derivative of x direction should be zero.

$$\frac{d}{ds}\left(\frac{1}{c(x)}\frac{dx}{ds}\right) = 0 \quad \Rightarrow \quad \frac{1}{c}\frac{dx}{ds} = Const. \quad \Rightarrow \quad \frac{\sin i}{C} = Const. \quad \Rightarrow \textit{Snell's law !!}$$

How does this angle i change in the direction of propagation?

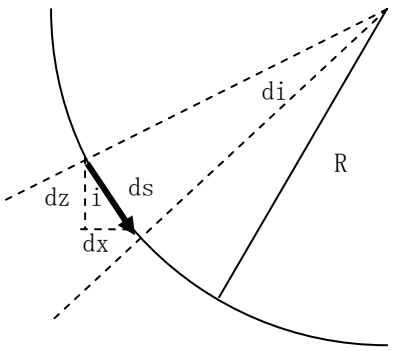
$$\frac{d}{ds}(\sin i) = \cos i \frac{di}{ds} = \frac{dz}{ds}\frac{di}{ds} = \frac{d}{ds}(pc) \qquad \Rightarrow \qquad \boxed{\frac{di(s)}{ds} = p\frac{dc}{dz}}$$

Therefore, the change of angle is related to the change of velocity.

If $\frac{dc}{dz}$ is large $\Rightarrow$ $\frac{di}{ds}$ is large

If $\frac{dc}{dz}$ is zero (c = const.) $\Rightarrow$ $\frac{di}{ds}$ is zero (i = const.) *Straight Ray !!*

RADIUS OF CURVATURE



R : the radius of curvature

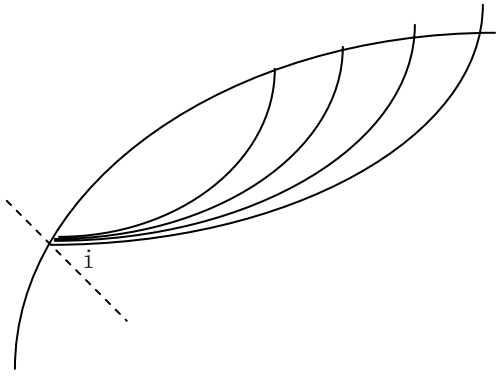
$$ds = Rdi$$

$$R = \frac{ds}{di} = \frac{1}{p}\frac{dz}{dc} = \frac{1}{p\left(\frac{dc}{dz}\right)} \quad \Rightarrow \quad \boxed{R = \frac{1}{p\left(\frac{dc}{dz}\right)}}$$

R is related to wavespeed gradient and ray parameter.

If $\frac{dc}{dz} = 0 \quad \Rightarrow R \to \infty$ *Straight Ray !!*

If $\frac{dc}{dz}$ large ⇒ rapid change in c *Strong Gradient*

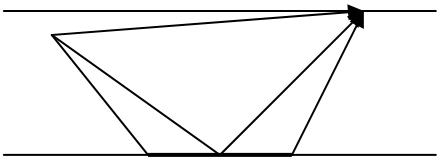


from $p = \frac{r \sin i}{c}$,

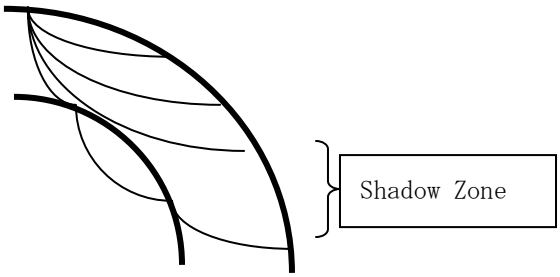
small i → small p → large R

AMPLITUDE-GEOMETRICAL SPREADING

Focusing-defocusing



Focusing effect



Defocusing effect

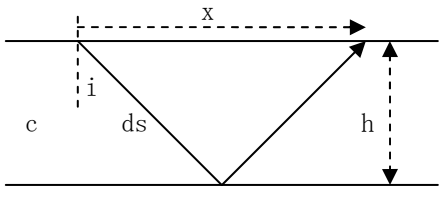
We examine the property of dp/dx

$$\frac{dp}{dx} = \frac{d}{dx}\left(\frac{dT}{dx}\right) = \frac{d^2T}{dx^2}$$

Small dx and large dp → dp/dx goes to infinity → large amplitude (focusing)

Large dx and small dp → dp/dx goes to zero → small amplitude (Shadow zone)

We also examine $x(p)$



$$T = 2\frac{ds}{c} \qquad \tan i = \frac{x/2}{h}$$

One layer : $x = 2h\tan i$

Multiple layers : $x = 2\sum_{j=0}^{n} h_j \tan i_j$

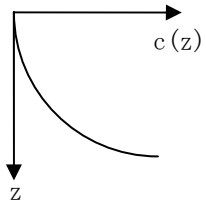
Continuous case

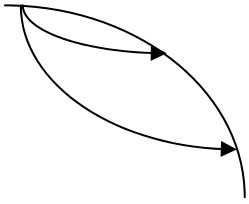
$$x(p) = 2\int_0^{z_p} \tan i\, dz = 2p\int_0^{z_p} \left(\frac{1}{c(z)^2} - p^2\right)^{-1/2} dz = 2p\int_0^{z_p} \frac{dz}{\sqrt{1/c^2 - p^2}} = 2p\int_0^{z_p} \frac{dz}{\eta}$$

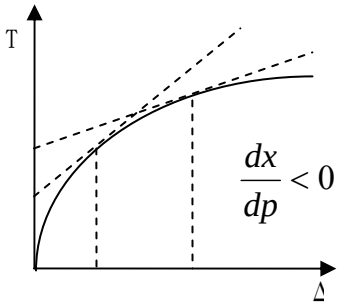
$$\frac{dx}{dp} = 2\int_0^{z_p} \frac{dz}{\sqrt{1/c^2 - p^2}} + 2p\frac{d}{dp}\left\{\int_0^{z_p} \frac{dz}{\sqrt{1/c^2 - p^2}}\right\} \Rightarrow \boxed{\frac{dx}{dp} \approx \left\{-\frac{1}{(dc/dz)_0}\right\} + \left\{+\int_0^{z_p} \frac{d^2c}{dz^2} dz\right\}}$$

The change of distance in terms of ray parameter is related to gradient of wave speed at surface and gradient of the change in wavespeed between surface and turning point.

Changes of velocity gradient, $\frac{d^2c}{dz^2}$, are small → large distance x for smaller ray parameter p, $\frac{dx}{dp} < 0$ → *"Normal"* or *Prograde behavior*







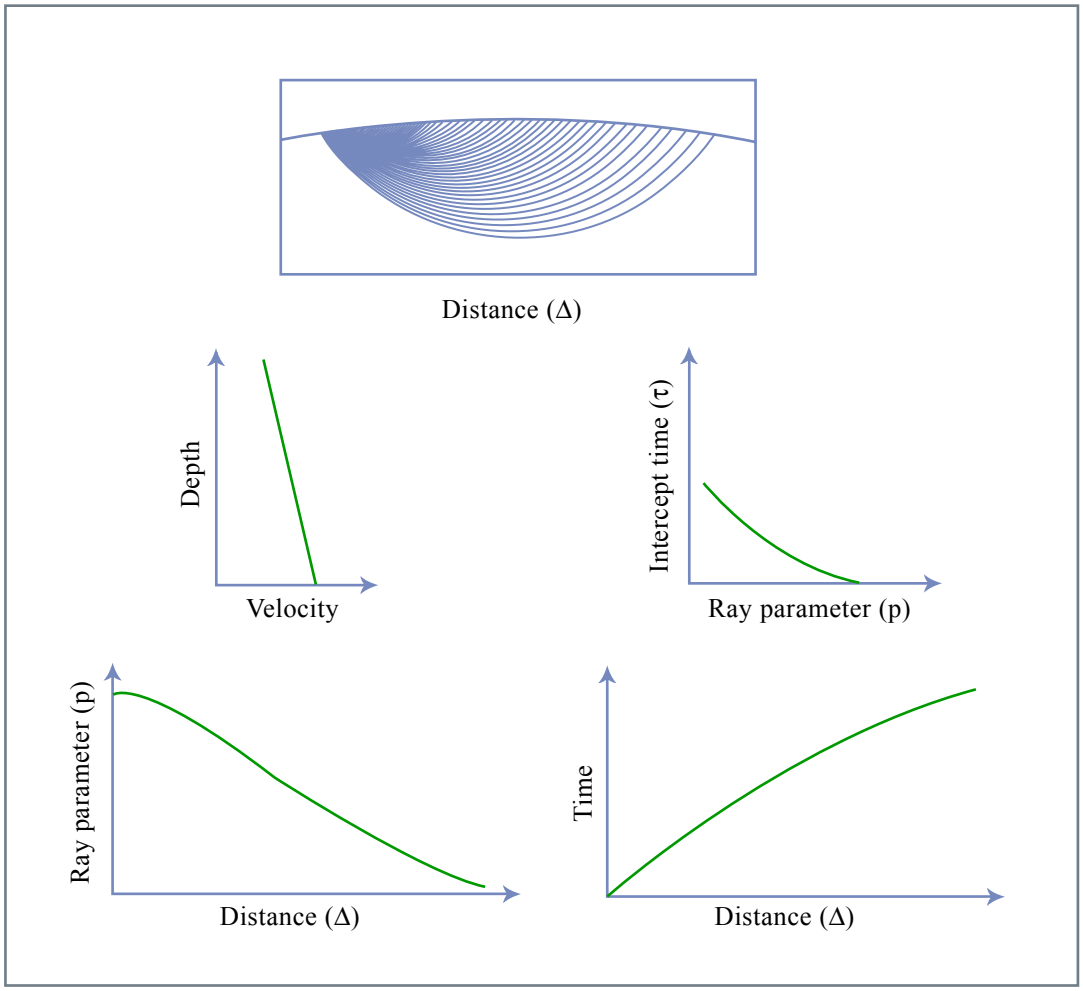


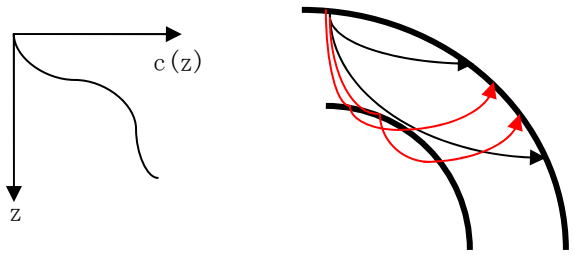
Figure by MIT OCW.

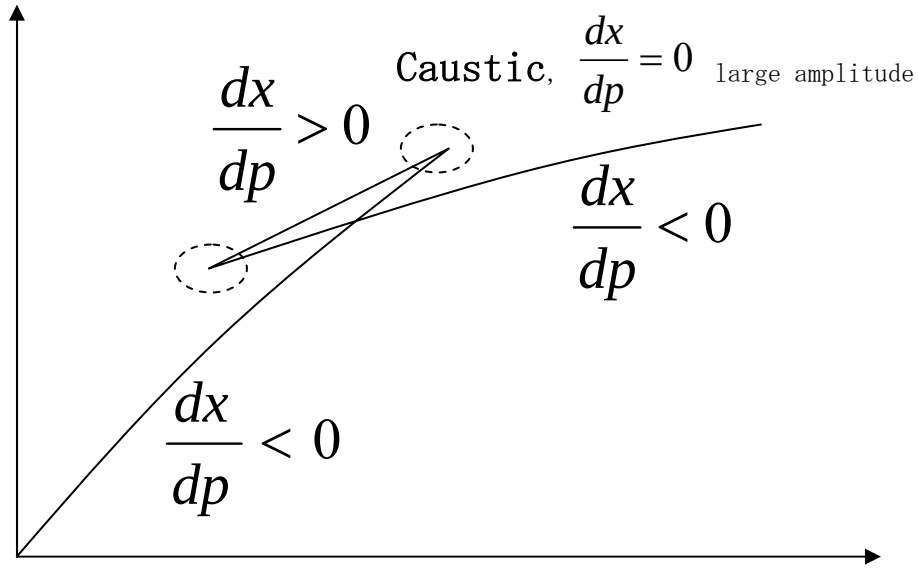
This figure represents ray paths, $T(\Delta)$, $p(\Delta)$, and $\tau(p)$ relationships for velocity increasing slowly with depth.

(Adapted from S. Stein and M. Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Sturcture*, Blackwell Publishing, p160)

Changes of velocity gradient, $\frac{d^2c}{dz^2}$, are large → samll distance x for smaller ray parameter p, $\frac{dx}{dp} > 0$ → *Retrograde behavior*

If $dp \neq 0$ and $dx = 0$ → $\frac{dx}{dp} = 0$ → *"Caustic"* or *focusing effect*





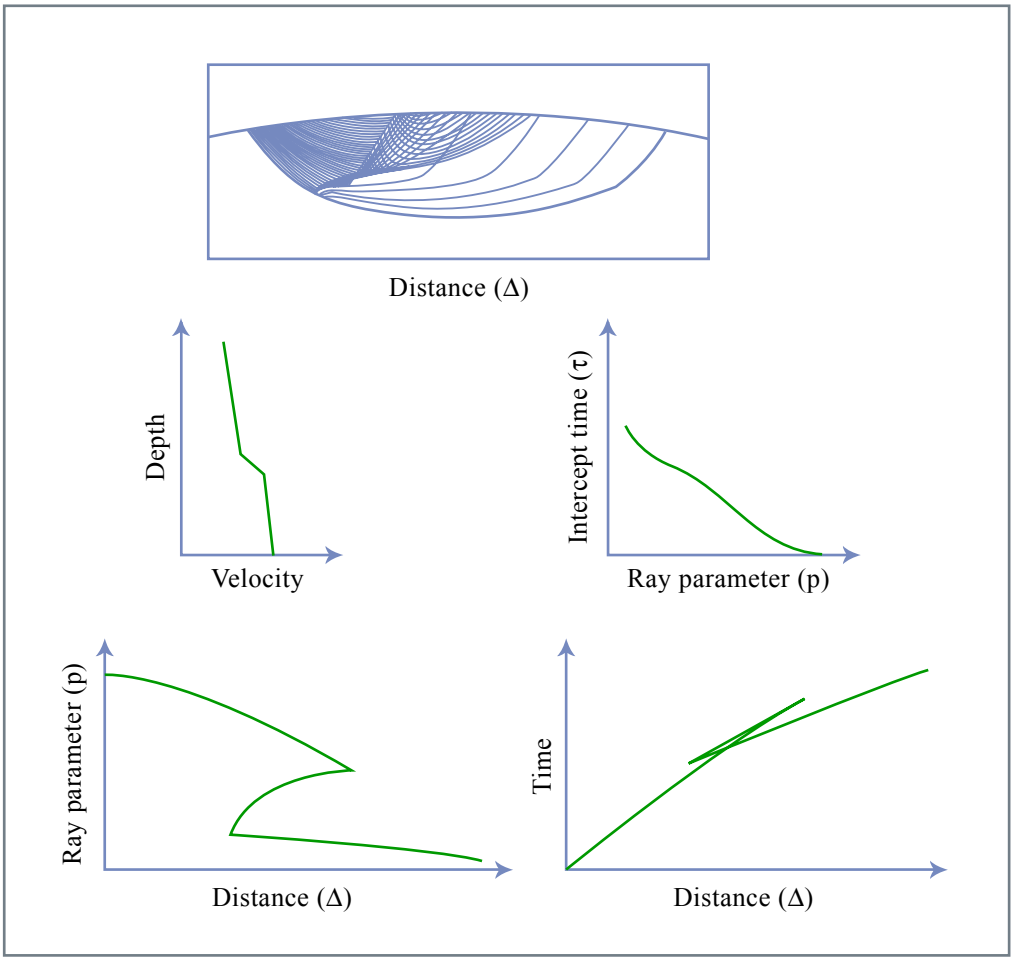


Figure by MIT OCW.

This figure represents ray paths, $T(\Delta)$, $p(\Delta)$, and $\tau(p)$ relationships for velocity increasing rapidly with depth. In this case we can see the triplication and retrograde behavior.

(Adapted from S. Stein and M. Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Sturcture*, Blackwell Publishing, p160)

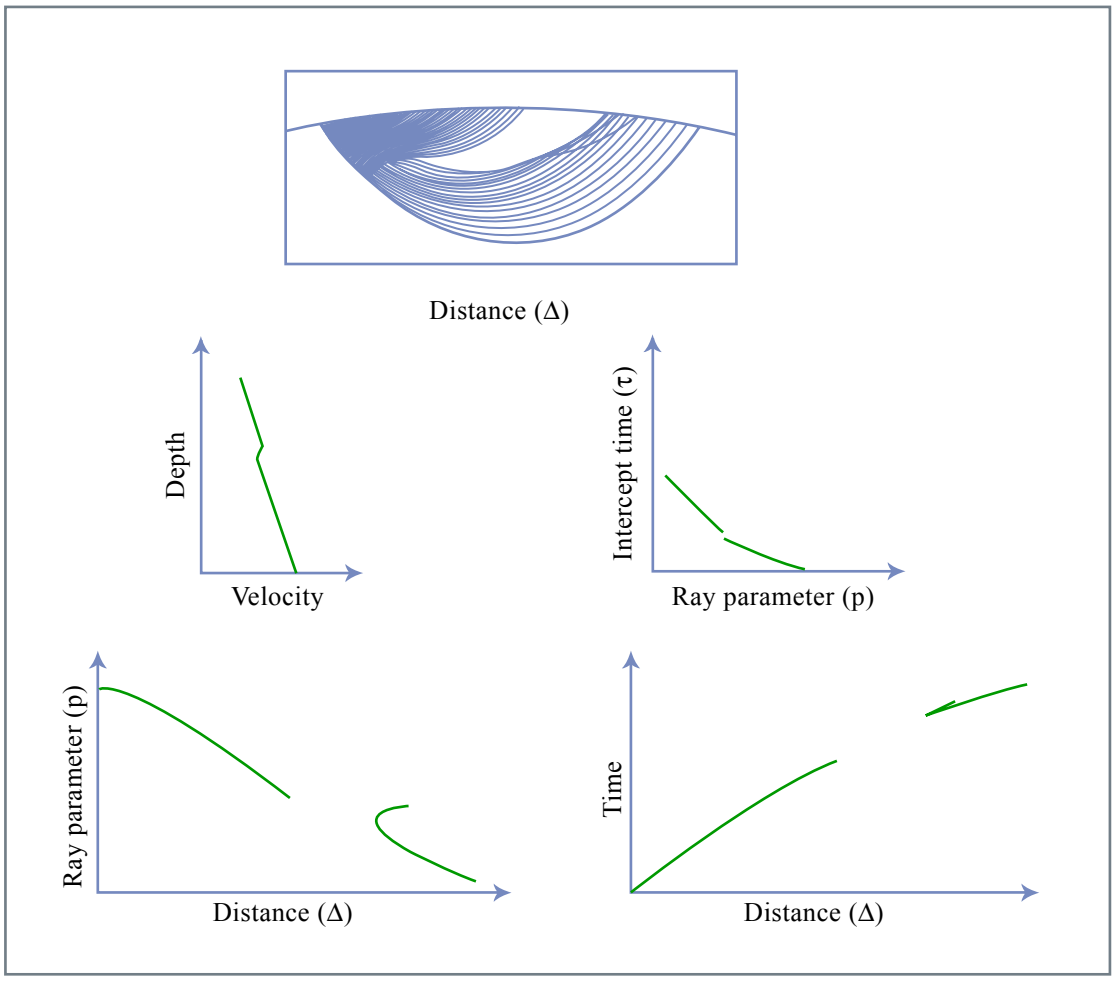
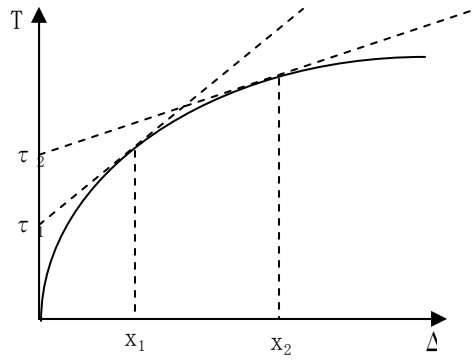


Figure by MIT OCW.

This figure represents ray paths, $T(\Delta)$, $p(\Delta)$, and $\tau(p)$ relationships for velocity decreasing slowly within a low-velocity zone. In this case we can see the shadow zone where no direct geometric arrivals appear, and hence discontinuous $T(\Delta)$, $p(\Delta)$, and $\tau(p)$ curves.

(Adapted from S. Stein and M. Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Sturcture*, Blackwell Publishing, p161)

$\tau - p$



$$T(p) = \tau(p) + \frac{dT}{dx}x = \tau(p) + px$$

$$\Rightarrow \qquad \tau(p) = T(p) - px$$

$$\Rightarrow \qquad \frac{d\tau}{dp} = -x(p)$$

The function τ(p) is called the *intercept-slowness* representation of the travel time curve. Just as p is the slope of the travel time curve, T(x), the distance x is minus the slope of the τ(p) curve.