$$(1)\ \Sigma F_i = m \cdot a_i \rightarrow \rho \frac{\partial u_i^2}{\partial t^2} = \sigma_{ij}\partial_j + f_i$$

Generalize Hooke's Law / Relationship between stress tensor & strain tensor

$$(2)\ \rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j (c_{ijkl} e_{kl}) = c_{ijkl}\partial j(e_{kl}) = (\lambda + \mu)\partial_i \frac{\partial u_k}{\partial x_l} + \mu \nabla^2 u_i$$

f_i=0 ↑ homogeneous media ↑ isotropic media

$$(i.e.,\ c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))$$

Using $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$ the above can be rewritten in a vector form:

$$(3)\ \ddot{\mathbf{u}} = \left(\frac{\lambda + 2\mu}{\rho}\right)\nabla(\nabla \cdot \mathbf{u}) - \left(\frac{\mu}{\rho}\right)\nabla \times (\nabla \times \mathbf{u})$$

We set $\theta = \nabla \cdot \mathbf{u}$, which is the cubic dilatation; and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, which is shear/rotation change, equation (3) can be rewritten as:

$$(4)\ \ddot{\mathbf{u}} = \alpha^2 \nabla\theta - \beta^2 \nabla \times \boldsymbol{\omega}$$

Our equations for *P*-wave and *S*-wave velocities:

$$(5a)\ \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$, or with $k = \lambda + 2/3\mu$, we can get $\alpha = \sqrt{\frac{k + 4/3\mu}{\rho}}$

$$(5b)\ \beta = \sqrt{\frac{\mu}{\rho}}$$

N.B. Equations (3) and (4) begin to look like wave equations, but we refer to them as equations of motion. Solving them for $\mathbf{u}$ is not easy, but can be done numerically. We first show that we can obtain wave equations for *P* and *S* waves. Then the solution is used to go back and solve the equation of motion. There are two ways of doing this.

1. (This is not done in Stein and Wysession's book) Isolate the part that only results in volume change and the part that only results in shear.
To isolate the volume component take the divergence:

$$\alpha^2 \nabla \cdot \nabla\theta - \beta^2 \nabla \cdot (\nabla \times \boldsymbol{\omega}) = \frac{\partial^2}{\partial t^2}(\nabla \cdot \mathbf{u}) = \frac{\partial^2}{\partial t^2}\theta$$

Because $\nabla \cdot (\nabla \times \boldsymbol{\omega}) = 0$, this becomes:

$$(6)\ \alpha^2 \nabla^2 \theta = \ddot{\theta}$$

To separate the shear part, take the curl:

$$\alpha^2 \nabla \times \nabla\theta - \beta^2 \nabla \times (\nabla \times \boldsymbol{\omega}) = \frac{\partial^2}{\partial t^2}(\nabla \times \mathbf{u}) = \frac{\partial^2}{\partial t^2}\boldsymbol{\omega}$$

Because $\nabla \times (\nabla\theta) = 0$, this becomes:

(7) $\beta^2 \nabla^2 \boldsymbol{\omega} = \ddot{\boldsymbol{\omega}}$

This is a simple way of isolating change in volume and change in rotation. A more elegant/powerful method is our second method:

2. Helmholtz decompositic of a vector field (☼)

(8) ☼$= \nabla A + \nabla \times \mathbf{B}$, with $\nabla \cdot \mathbf{B} = 0$

B is the vector potential, and A is the scalar potential.

There are 4 components: 1 gradient and 3 curl. NEED the constant that $\nabla \cdot \mathbf{B} = 0$.

Examples:

Gravity $g = -\nabla U_{grav}$ rotation free because gravity is a conservative force field

Magnetic Field $\mathbf{B} = -\nabla V_{mag} + \nabla \times \mathbf{f}$ e.g. Lorentz force → Maxwell's equations

Here we can write for displacement:

(9) $\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}$, with $\nabla \cdot \boldsymbol{\psi} = 0$

This is the Helmholtz representation with displacement potentials ϕ and $\boldsymbol{\psi}$.

This can be substituted into the equations of motion to get:

(10) $$\nabla\left\{\left(\frac{\lambda + 2\mu}{\rho}\right)\nabla^2\phi - \ddot{\phi}\right\} + \nabla \times \left\{\left(\frac{\mu}{\rho}\right)\nabla^2\boldsymbol{\psi} - \ddot{\boldsymbol{\psi}}\right\} = 0$$

One way to solve this equation is to say that both parts are zero at the same time.

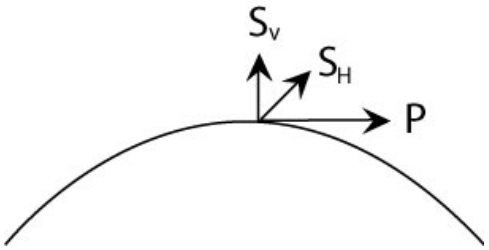
The *P*-wave potential equation:

(11) $$\ddot{\phi} = \frac{\lambda + 2\mu}{\rho}\nabla^2\phi$$

The *S*-wave potential equation:

(12) $$\ddot{\boldsymbol{\psi}} = \left(\frac{\mu}{\rho}\right)\nabla^2\boldsymbol{\psi}$$

These are not the only solutions to (10). There could be a coupling between the *P*-wave and *S*-wave parts at low frequencies.



Instead of vector wave equation we can write two scalar wave equations:

(13a, b) $\ddot{\psi}_{SV} = \beta^2 \nabla^2 \psi_{SV}$, where SH(SV) denotes the horizontally (vertically) polarized shear wave.
$\ddot{\psi}_{SH} = \beta^2 \nabla^2 \psi_{SH}$

(14) $\nabla \times \boldsymbol{\psi} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \psi_x & \psi_y & \psi_z \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$

In 3-D:

$$u_x = \frac{\partial \phi}{\partial x} + \left(\frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z} \right)$$

$$u_y = \frac{\partial \phi}{\partial y} - \left(\frac{\partial \psi_z}{\partial x} - \frac{\partial \psi_x}{\partial z} \right)$$

$$u_z = \frac{\partial \phi}{\partial z} + \left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} \right)$$

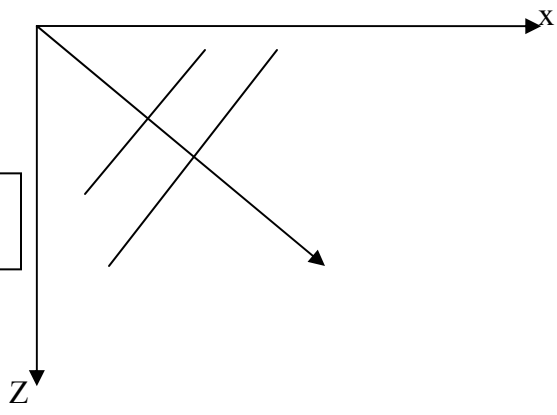
In 2D, we consider propagation in x-z plane, $\frac{\partial}{\partial y} = 0$

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi_y}{\partial z}$$

$$u_y = \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x} \rightarrow$$

There is displacement in the y-direction but NO propagation!!

$$u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi_y}{\partial x}$$



u_x → P+S_v, coupling, ϕ and $\boldsymbol{\psi}$

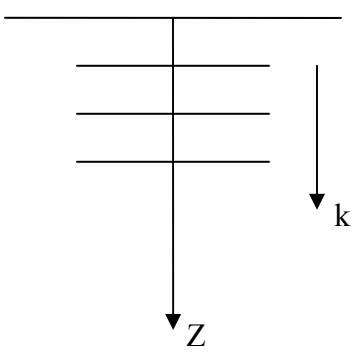
u_y → pure shear S_H, only $\boldsymbol{\psi}$

u_z → P+S_v, coupling, ϕ and $\boldsymbol{\psi}$

In 1D, just vertical propagation:

$$u_x = -\frac{\partial \psi_y}{\partial z}, u_y = \frac{\partial \psi_x}{\partial z}, u_z = \frac{\partial \phi}{\partial z}$$

u_x → pure shear
u_y → pure shear
u_z → pure *P*-wave



Recall:

(5a, b)
$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{k + 4/3\mu}{\rho}}$$
$$\beta = \sqrt{\frac{\mu}{\rho}}$$

where α is the *P*-wave velocity and β is the *S*-wave velocity. The bulk modulus (k) can be measured in the lab, hence, it is sometimes useful to write the velocities in terms of this constant. The acoustic sound wave speed can be written as:

(15) $\sqrt{\frac{k}{\rho}}$

Bulk sound speed: $V_\phi^2 = \alpha^2 - 4/3\,\beta^2 = \frac{k}{\rho}$

Even though it is not physical wave elastic theory can derive it.

Poisson's medium ($\lambda=\mu$)

$\alpha = \sqrt{3}\beta$ This will work for most areas of the mantle…this is not used generally in "modern papers"

Solutions to the Wave Equation

$$\ddot{\phi} = \alpha^2 \nabla^2 \phi$$
$$\ddot{\Psi}_{SV} = \beta^2 \nabla^2 \Psi_{SV}$$
$$\ddot{\Psi}_{SH} = \beta^2 \nabla^2 \Psi_{SH}$$

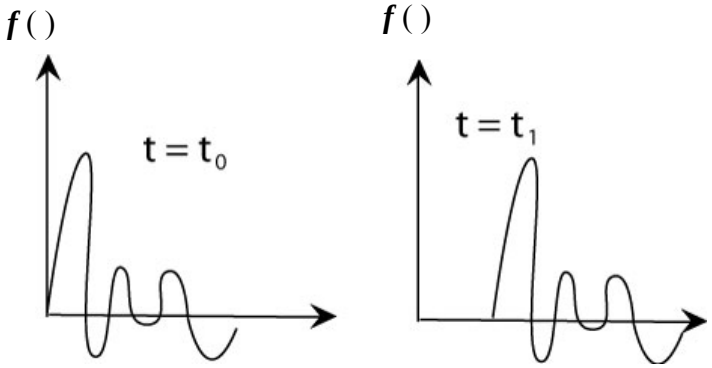
1. d'Alembert's solution

f(± x ± ct),

$\phi(x,t) = f(x - ct) + g(x + ct)$, *f* and *g* are arbitrary functions as long as they are at least twice differentiable at time and space.

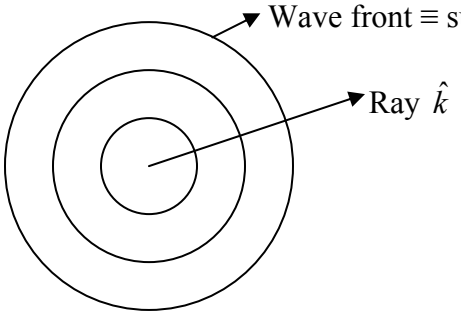
(x-ct) propagation in the positive x-direction
(x+ct) propagation in the negative x-direction
c is the wave speed or phase velocity

$c = \frac{x_1 - x_0}{t_1 - t_0}$ looks at how long the peak propagates through the system



Example of a plane wave propagation in positive x-axis; function ***f*** ( ) remains same if argument (the phase!) remains same; this happens if x increases when t increases → motion in positive x axis.

Plane Waves: Surfaces of constant phase change → wave front



One example of a general solution is the harmonic function:

$A\exp(i(x-ct)) = A(\cos(x-ct)+i\sin(x-ct))$

$x-ct = phase = \varphi$

$so...Ae^{i\varphi}$

The real part of the solution gives the displacement; the complex part can be used to describe amplitude decay; makes the notation elegant.

Angular frequency: $\omega = kc$, dispersion relationship $k = \omega/c$

Wave length: $\lambda = cT = 2\pi/k$

Wave number: $k = 2\pi/\lambda$, high wave number corresponding to short wave length

$\psi=(kx-\omega t)$ a.k.a. Fourier duals will transform from time to frequency domain in 3D $(\mathbf{k}\cdot\mathbf{x}-\omega t)$…

_P_-wave and _S_-wave potential

The scalar potential for a harmonic plane *P*-wave satisfying motions of equation (6) is:

$\phi(\mathbf{x},t) = A\exp(i(\mathbf{k}\cdot\mathbf{x}-\omega t))$

The vector potential for S-wave satisfying equation (7) is:

$\mathbf{\Psi}(\mathbf{x},t) = \mathbf{B} \times \hat{\mathbf{k}} \exp(i(\mathbf{k} \cdot \mathbf{x} - \omega t))$, where $\mathbf{B} = (B_x, B_y, B_z)$

With this in mind interesting to look at our 1D example from above where $\mathbf{k} = (0,0,k_z)$

So the resulting displacement for *P*-wave is the gradient

$\mathbf{u}(z,t) = \nabla\phi = (0,\ 0,\ \ ik_z)A\exp(i(k_z z - \omega t))$

which has a nonzero component only along the propagation direction z.

The resulting displacement for *S*-wave is the curl

$\mathbf{u}(z,t) = \nabla \times \mathbf{\Psi} \quad = (ik_z B_x,\ ik_z B_y, 0)\exp(i(k_z z - \omega t))$

whose component along the propagation direction z is zero.

Separation of variables

$\ddot{\phi} = \alpha^2 \nabla^2 \phi$

$\phi = \phi(x,t) = \phi(x,y,z,t)$

$\phi = X(x)Y(y)Z(z)T(t)$

if you plug the bottom into the wave equation and divide by *XYZT*, you get:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{C^2}\frac{d^2T}{Tdt^2} = 0$$

And then we can set each term as a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2, \frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2 \text{ spatial parts;}$$

$$\frac{1}{C^2}\frac{d^2T}{dt^2} = -\omega^2 \text{ temporal part;}$$

$$k_x^2 + k_y^2 + k_z^2 - (\frac{\omega}{c})^2 = 0 \text{ dispersion relationship.}$$

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \rightarrow \exp(\pm ik_x x) = X(x)$$

$\phi = \exp(i(\pm k_x x \pm k_y y \pm k_z z \pm \omega t))$

$\phi = \exp(i(\pm \mathbf{k} \cdot \mathbf{x} \pm \omega t))$

k is the propagation direction and ***x*** is the position vector

The third way you can find the solution is taking a Fourier transform.

Notes: Patricia M Gregg (Feb 16, 2005)