STRESS and STRAIN

Reading: Section 2.3 of Stein and Wysession.

In this section, we will see how Newton's second law and Generalized Hooke's law can be used to characterize the response of continuous medium to applied forces. The stress tensor describes the forces acting on internal surfaces of a deformable continuous medium. The strain tensor describes the distortion of (or the variation in displacement within) the body. Stress alone will not cause wave motion, but the equation of motion describes how spatial variations in stress produce acceleration.

Stress (σ) is related to strain (ε) by:

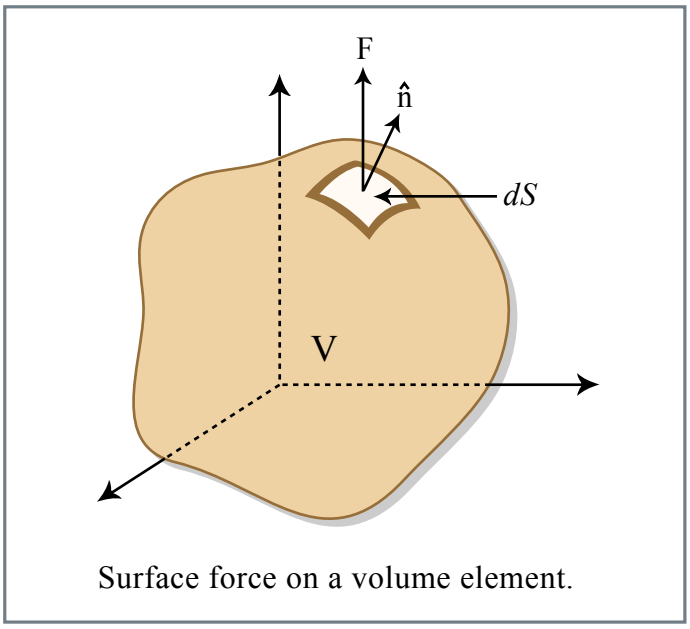
In 1D: $\sigma = E\varepsilon$ where E is Young's modulus (NBE has dimension of stress).

Stress

We consider two types of force that can act on a body:

1. Body force (f_i) – this force acts everywhere in the body and the net force is proportional to the volume of the body. E.g., gravity, electro-magnetics.
2. Surface force (T_i) – this force acts on the surface of a body and the net force is proportional to the surface area of the body.

Consider the forces acting on a small volume, V, with surface S, within a large continuous medium.



Surface force on a volume element.

Figure by MIT OCW.

(Figure 1, Adapted from *Stein & Wyssession,* 2003)

$Stress = \frac{F}{A}$, or force per unit area (N/m^2), 1 N/m^2 = 1 Pa (Pascal)

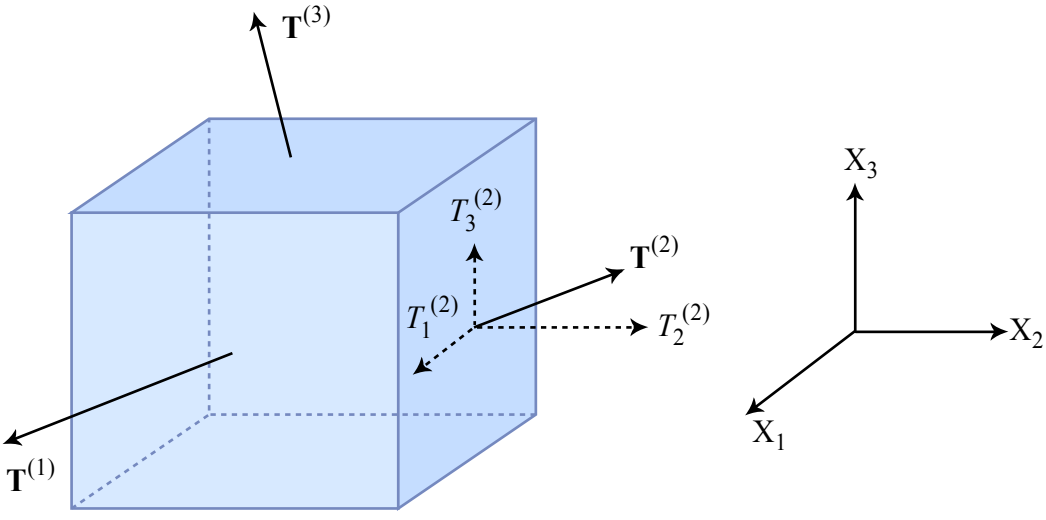
1atm= 1000mbar = 1 bar = 10^5Pa (force equivalent to the apple sauce from one apple spread over $1m^2$)

Examples:
410km ~ 14 GPa
660km ~ 23 GPa
CMB ~ 135 GPa
Center of the earth ~ 360 GPa

The surface force **F** acts on each element of surface **dS** which has a unit normal vector**, n**. The forces acting on the surfaces of a volume element can be described by three traction vectors. By definition the traction vector, **T**, is the limit of the surface force per unit area at any point as the area becomes infinitesimal.

Stress vector = Traction = $\mathbf{T} = \lim_{\delta S \to 0} \frac{\mathbf{F}}{\delta S} = (T_1, T_2, T_3)$

Each traction vector acts on a surface perpendicular to a coordinate axis: $\mathbf{T}^{(i)}$ is the traction vector working on the surface with its unit normal **n** in direction *I* (Fig. 2).



Traction vectors on the faces of a volume element.

Figure by MIT OCW.

Fig. 2 (Adapted from *Stein & Wyssession*, 2003)

REMEMBER: we have made the assumption/approximation that the medium is a continuum. For a more detailed approach refer to L. E. Malvern, 1969, *Introduction to the Mechanics of a Continuous Medium*, Englewood Cliffs, N. J., Prentice-Hall.

Equation of motion

Using Newton's second low we can write the equation of motion:

$$\sum F_i = ma_i$$

$$\sum F_i = \text{body forces} + \text{surface forces} = f_i + T_i = ma_i = m\frac{\delta^2 u_i}{\delta t^2}$$

or, see Fig. 3:

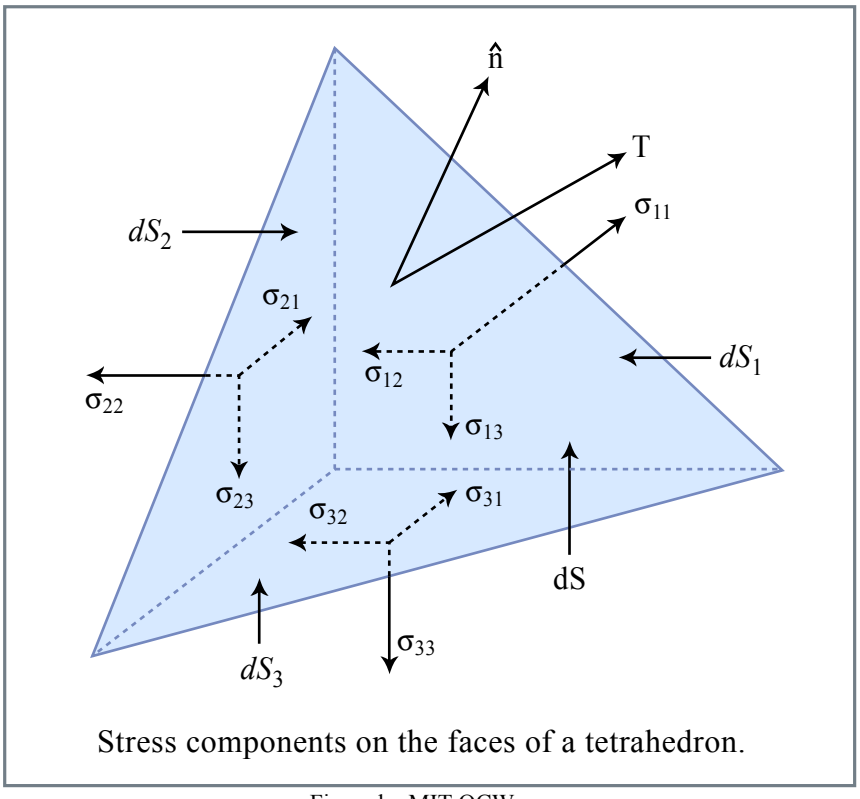
$$\sum F_i = T_i dS - \sigma_{1i} dS_1 - \sigma_{2i} dS_2 - \sigma_{3i} dS_3 + f_i dV = \rho \frac{\partial^2 u_i}{\partial t^2} \partial V .$$

As the internal surfaces are, in general, not known we need to get an expression independent of dS or dS_i. This can be done by realizing that dS_i is the orthogonal projection of surface dS along axis (i): $dS_i = \cos\varphi_i dS = n_i d$, with φ_i the angle between the normal vector **n** and T_i (see Fig. 3). If we assume that the system is in equilibrium (i.e., $a_i=0$) and that dV/dS goes to zero, then the equation above is equal to zero and division by dS gives:

$$T_i = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 = \sigma_{ji} n_j$$

$$\sigma_{ji} = T_i^{(j)}$$

$$\sigma_{ji} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \end{pmatrix} = \begin{pmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{pmatrix}.$$



Stress components on the faces of a tetrahedron.

Figure by MIT OCW.

Fig. 3 (Adapted from *Stein & Wyssession,* 2003)

In the absence of body forces, the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$), therefore there are only 6 independent elements. The diagonal elements represent the normal stress and the off- diagonal elements the shear stress. A symmetric tensor can also be diagonalized; in this case that means that the body can be rotated such that the tractions become parallel to the normals to the surfaces being investigated.

In other words,
$T_i = \sigma_{ji} n_j = \sigma_{ij} n_j = \lambda n_i$

This is an eigenvalue/eigenvector problem:

$$\left(\sigma_{ij} - \lambda \delta_{ij}\right) n_j = 0$$
$$\left(\boldsymbol{\sigma} - \lambda \mathbf{I}\right) \mathbf{n} = \mathbf{0}$$

Where λ is the eigenvalue or principle stresses and **n** is the eigenvector or principle axis.

$$\det \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0.$$

Taking the determinant of the stress tensor gives a cubic equation for λ and three solutions which can be plugged back in to give three eigenvectors, the principal stress axes. Digitalization makes all the shear components disappear and the remaining diagonal components are the principle stresses.

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Where the magnitude of the components are: $|\sigma_1| \rangle |\sigma_2| \rangle |\sigma_3|$

Types of stress:

1. Uni-axial stress – e.g., $\sigma_1=0$, $\sigma_2=0$, $\sigma_3 \neq 0$
2. Plane stress– e.g. $\sigma_1 \neq 0$, $\sigma_2=0$, $\sigma_3 \neq 0$
3. Pure shear – this is an example of plane stress where two of the stress components are equal in magnitude but opposite in direction: e.g., $\sigma_1 = -\sigma_3$
4. Isotropic, lithostatic, hydrostatic stress – same stress everywhere, where pressure, or mean stress, *M*:

$$M = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)$$

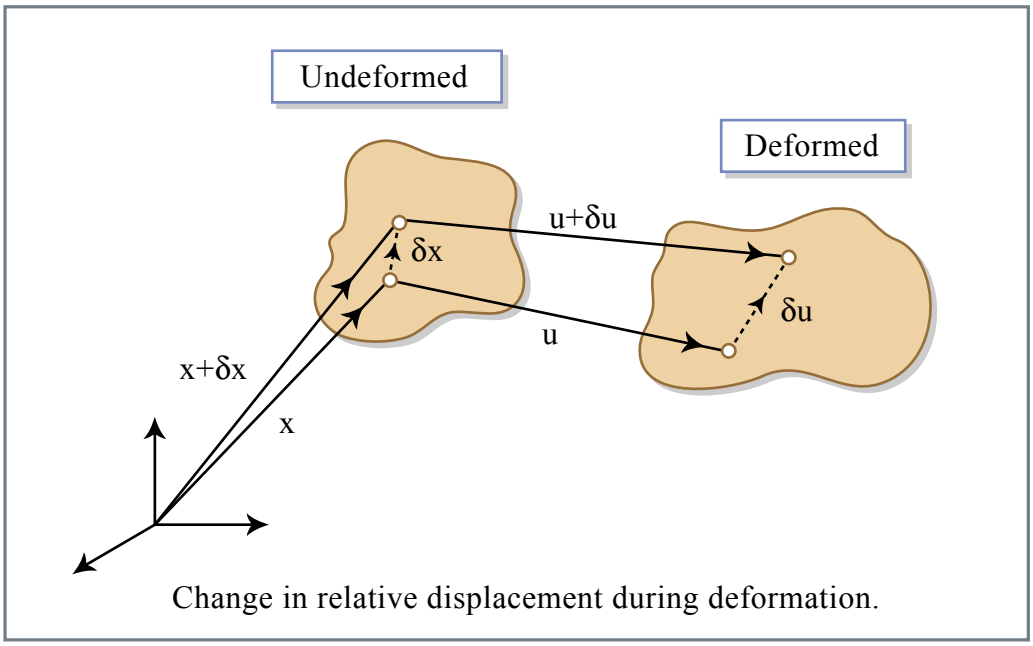
5. Deviatoric stress – defined as the remaining stress state after the effect of the mean stress has been removed. The deviatoric stress gives rise to motion. This is the most important stress in seismology.

$$D_{ij} = \sigma_{ij} - M\delta_{ij}$$

$$D = \begin{pmatrix} \sigma_{11} - M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - M \end{pmatrix}.$$

Strain

When stress is applied to a non-rigid body deformation occurs. This deformation can be described by the strain tensor. Strain is a relative measurement and is therefore dimensionless.



Change in relative displacement during deformation.

Figure by MIT OCW.

(Adapted from *Stein & Wyssession*, 2003)

In 1D the strain is given by: $\varepsilon_{xx} = \dfrac{u(x+\delta x) - u(x)}{\delta x} = \dfrac{\delta u}{\delta x} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial x}\right)$, where we used the linearization: $u(x+\delta x) \cong u(x) + \delta u(x)$. This is justified as long as the change in displacement is smooth over a distance δx:

$\delta u \approx \int_x \dfrac{\partial u}{\partial x}\delta x$ (infinitesimal strain theory).

$$\mathbf{u}(x) = \mathbf{u}(x_0) + \frac{\partial u}{\partial x}\mathbf{d}$$

$$\mathbf{u}(x) = \mathbf{u}(x_0) + \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \\ & etc & \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \mathbf{u}(x_0) + \mathbf{J}\mathbf{d}$$

Where **J** is the Jacobian transformation tensor.

$$\mathbf{J} = \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$$

ε is the symmetric matrix strain tensor ε_{ij}

$$e_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

Ω is the antisymmetric matrix rotation tensor Ω_{ij}

$$\Omega_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}(\ ) \\ -\frac{1}{2}(\ ) & -\frac{1}{2}(\ ) & 0 \end{pmatrix}$$

In seismology we are interested only in the distortion of the material (strain tensor) and not the rigid body rotation (rotation tensor).

The trace (tr) of the strain tensor is
$tr(\boldsymbol{\varepsilon}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \nabla.\mathbf{u}$, which is also known as the cubic dilatation (Θ).
Divergence of the displacement field relates to the relative change in volume. The trace of the rotation vector is zero, i.e. a rigid body rotation does not involve a volume change.

References

Stein, S. & M. Wysession, 2003, *An Introduction to Seismology, Earthquakes, and Earth Structure*, Malden, M.A., Blackwell Publishing Ltd.