**12.510: Introduction to Seismology** March 7, 2005

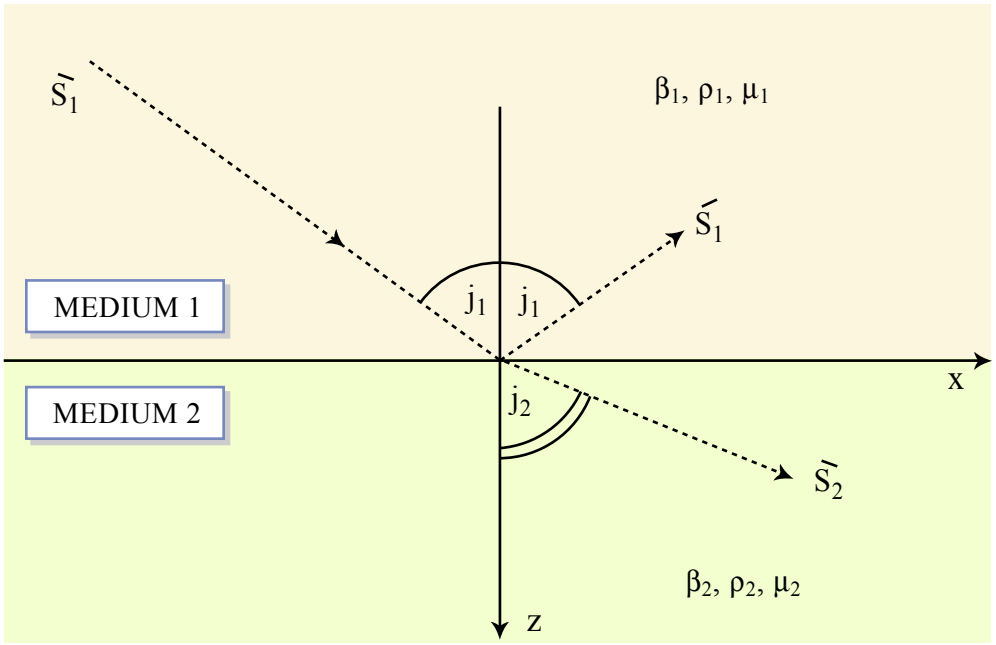
# Lecture 9

Todays topics:

- Critical and post critical reflection
- Phase shifts and pulse distortion
- Layer over half space

## I. Critical and Post Critical Reflection

Last lecture we considered an SH field:




SH wave incident on a solid-solid boundary.

Figure by MIT OCW.

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, fig. 2.6-2, p. 76, Blackwell Publishing.

We know from Lecture 7 that for a post critical wave:

$$\eta_2 = i\hat{\eta}_2 \tag{1}$$

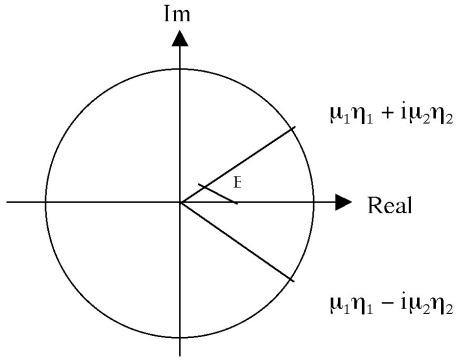
and for a reflected wave,

$$\frac{\overset{/}{A_1}}{\underset{\backslash}{A_2}} = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2} \tag{2}$$

Combining the two, the reflection coefficient for a post critical wave is given by

$$R = \frac{\mu_1\eta_1 - i\mu_2\hat{\eta}_2}{\mu_1\eta_1 + i\mu_2\hat{\eta}_2} = e^{i2E} \tag{3}$$

Note that equation (3) is a complex number divided by its complex conjugate. The complex values can be visualized in the complex plane.




From this figure we can see that

$$E = tan^{-1}(\frac{\mu_2\hat{\eta}_2}{\mu_1\eta_1}) \tag{4}$$

which shows that the postcritical reflected wave will undergo a phase shift. $|R| = 1$, and the value of the phase shift is dependent on the angle of incidence.

The phase shift can be found by comparing the potentials of the incident and reflected waves:

$$\overset{/}{\phi_1} = R\ \overset{\backslash}{\phi_1} = e^{i2E}\ \overset{\backslash}{\phi}$$

$$= e^{i2E}\ \overset{\backslash}{A_1}\ e^{i\omega(px-\eta_1 z-t)}$$

$$= \overset{\backslash}{A_1}\ e^{i[2E+\omega(px-\eta_1 z-t)]}$$

$$= \overset{\backslash}{A_1}\ e^{i[\omega(\frac{2E}{\omega}+px-\eta_1 z-t)]}$$

$$= \overset{\backslash}{A_1}\ e^{i\omega[px-\eta_1 z-(t-\frac{2E}{\omega})]} \tag{5}$$

Also recall that transmitted postcritical waves become evanescent:

$$\overset{\backslash}{\phi_2} = T\ \overset{\backslash}{\phi_1} = \overset{\backslash}{A_2}\ e^{-\omega\hat{\eta}_2 z}e^{i\omega(px-t)} \tag{6}$$

with the $e^{-\omega\hat{\eta}_2 z}$ term determining exponential decay with depth.

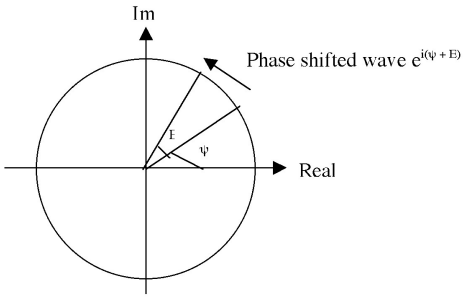
Equation (5) shows that in a postcritical reflected wave, there is a phase shift of $\frac{2E}{\omega}$.

## II. Phase Shift

Several things can be noted about the phase shift:

1. It is frequency-dependent
2. As $\omega \to \infty, E = tan^{-1}(\frac{\hat{\eta}_2 \mu_2}{\eta_1 \mu_1}) \to 0$

Thus, at very high frequencies, no phase shift occurs. We can visualize the phase shift using the complex plane. Consider a phase-shifted wave $e^{i(\psi+E)}$:




Remember that

$$e^{i\psi} = cos\psi + isin\psi \tag{7}$$

If $E = 0$, no shift occurs. If $E = \pi$,

$$cos(\psi + \pi) + isin(\psi + \pi) = -cos\psi - isin\psi = -e^{i\psi} \tag{8}$$

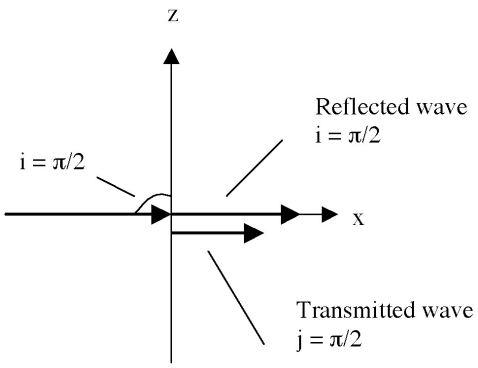
so a phase shift of $E = \pi$ produces a polarity flip.

Now, consider the case for $i = i_c$. At the critical angle, things are generally "well-behaved," e.g.

$$\eta_2 = 0 = \hat{\eta}_2 \tag{9}$$

Equation (9) implies $E = 0$, which shows there is no phase shift at $i = i_c$. E slowly increases as i increases to values greater than $i_c$.

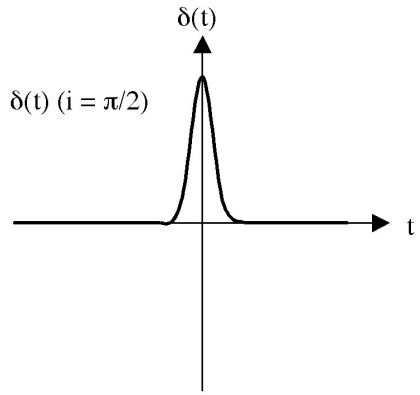
Eventually, i increases to $\frac{\pi}{2}$ (grazing incidence):



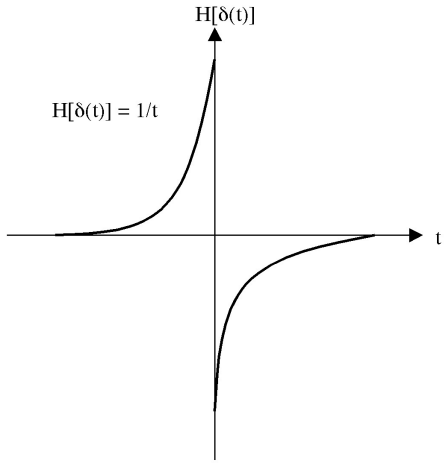
In this case, the vertical slowness disappears, i.e. $\eta_1 \to 0$.
A phase shift of $\frac{\pi}{2}$ is known as a **Hilbert Transform**. For instance, if there is a function $f(t)$ that describes a signal,

$$H[f(t)] = f(t) \otimes (-\frac{\pi}{t}) \tag{10}$$

is the Hilbert Transform of that signal. For example, if you have an impulse $\delta(t)$ with a 90 degree incidence,




the graph of the Hilbert Transform for $\delta(t)$ is given by

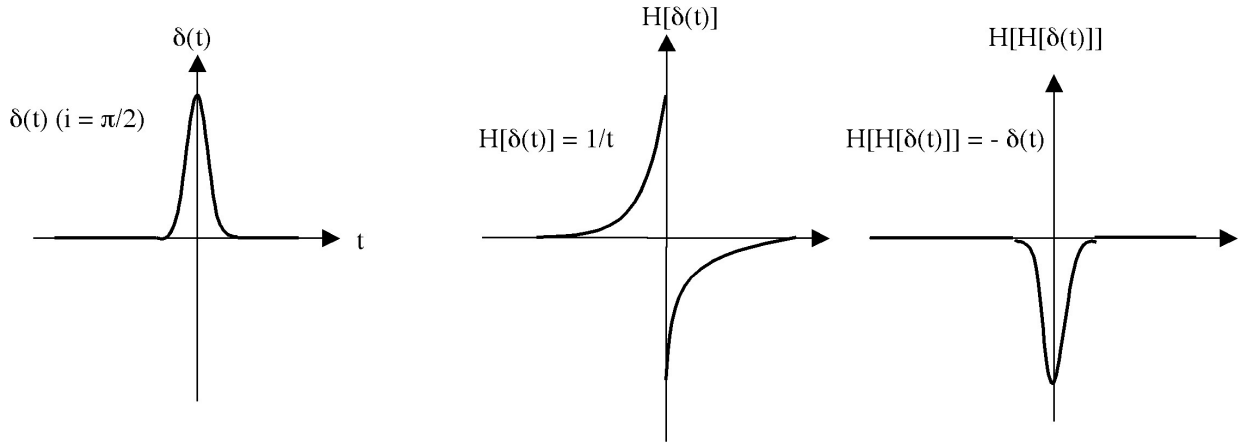



Any phase shift between $\psi = 0$ and $\psi = \pi$ can be explained by a linear combination of the incoming wave ($f(t)$) and its Hilbert Transform. In other words, consider a phase shift of E, such that $0 < E < 180$,

$$f'(t) = f(t)cosE + H[f(t)]sinE \tag{11}$$

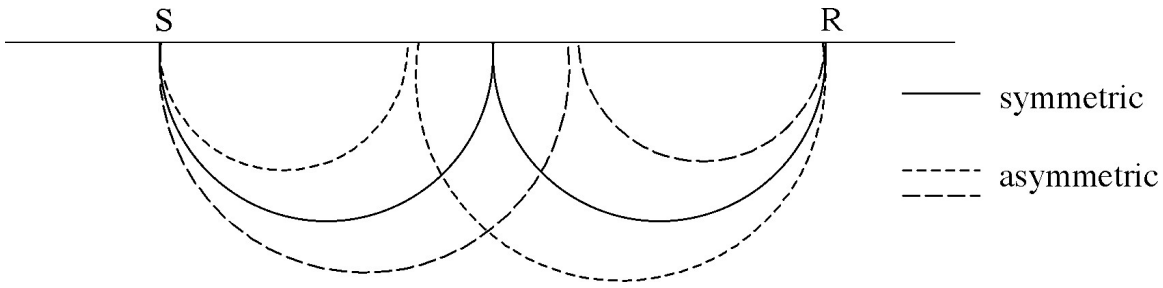
When $E = 0$ ($i = i_c$), $sinE = 0$, so $f'(t) = f(t)$. When $E = \pi$, $f'(t) = -f(t)$. When $E = \frac{\pi}{2}$, $f'(t) = f(t)$.

Additionally, a Hilbert Transform on the Hilbert Transform of impulse $\delta$ ($H[H[f(t)]]$, where $f(t) = \delta(t)$) results in $-f(t)$.




Also, consider a source impulse in the earth. If an impulse $\delta$ leaves a source, each instance of reflection provides a $\frac{\pi}{2}$ phase shift (in the 2-D case...the 3-D case provides a phase shift of $\pi$). See the Shearer text for more information on the Hilbert Transform.

Now, let's consider the case where a ray bundle is shot from a source on the surface of the earth. A variety of PP waves will travel to the receiver, but only one of them is symmetric. In the earth, wave speed increases with depth. Therefore an asymmetric PP wave will have a shorter path within a low-velocity medium and a longer path within a high-velocity medium, thus arriving at the receiver earlier than the symmetric PP wave with which we are concerned. In other words, all of the source energy arrives before the actual PP wave.




With a frequency ($\omega$) approaching infinity, the phase shift approaches zero, and the arrival of the PP wave is obvious. However, in the case of a finite frequency, the arrival of the wave is at the peak of the Hilbert Transform curve. Consequently, in order to work with these types of waves, it is necessary to have either $\omega \to \infty$ or to use synthetic seismograms, otherwise large mistakes can be made in the wave analysis. The only way to properly identify waves with finite $\omega$ is to calculate what the waveforms should look like.

## III. Layer Over Half-Space

We will now look at travel time curves for 3 types of waves through a homogeneous medium:

A. Direct Wave
B. Reflected Wave
C. Head Wave

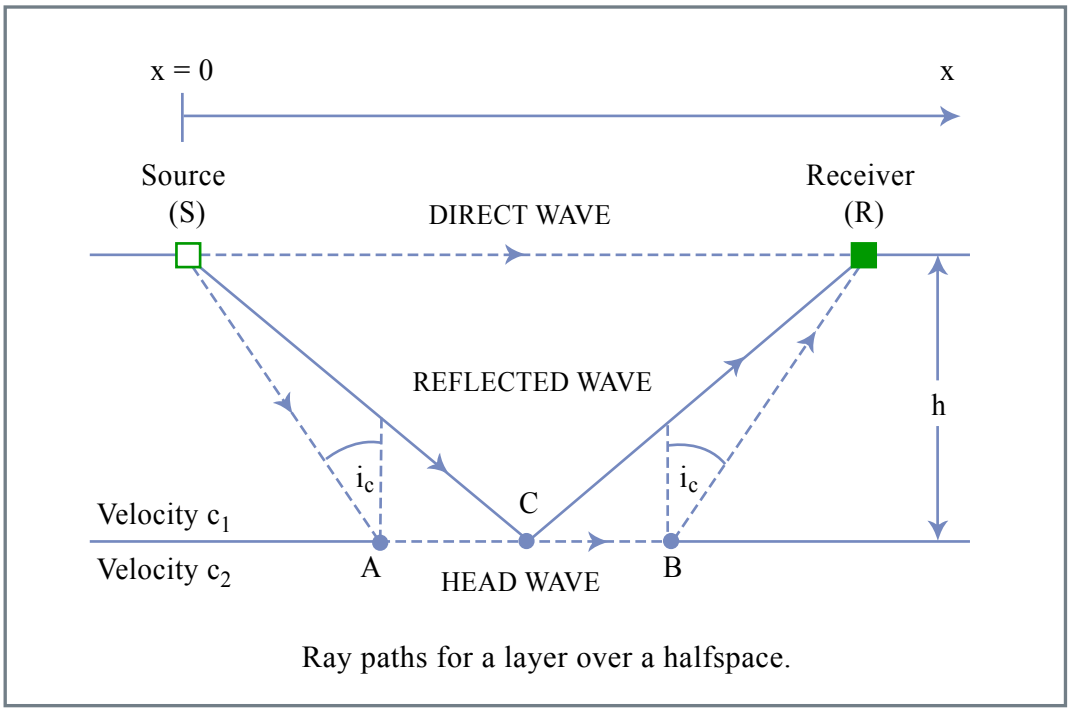



Figure by MIT OCW.

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, fig. 3.2-1, p. 120, Blackwell Publishing

A. Direct Wave

The travel time for a direct wave is very straightforward, as the wave travels along the most direct path from the source to the receiver.

$$t_a = \frac{x}{c_1} \tag{12}$$

Thus, the travel time curve can be plotted:

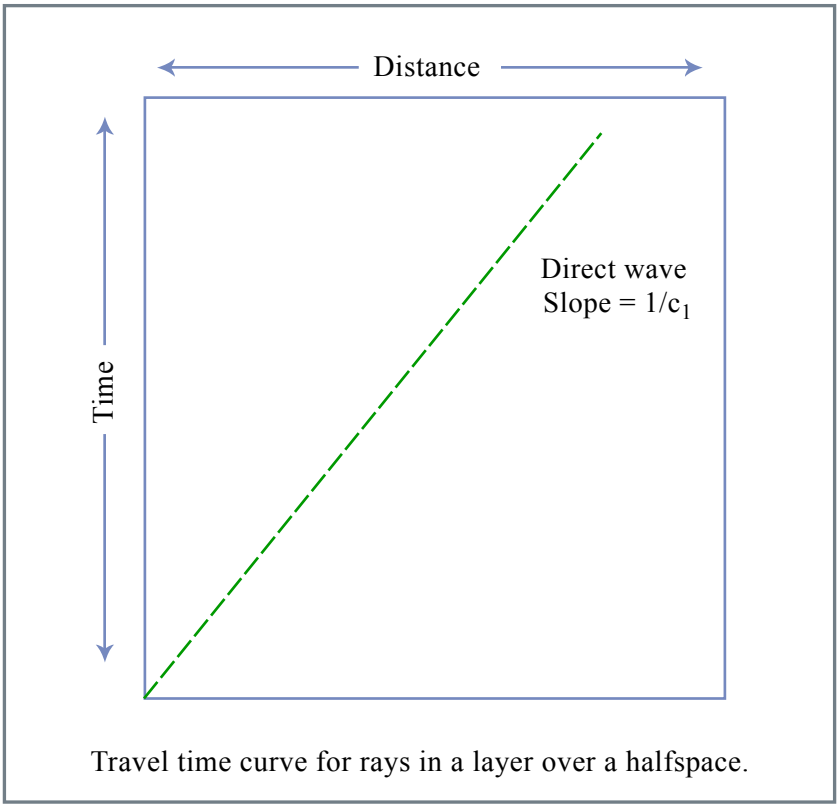



Figure by MIT OCW.

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, fig 3.2-2, p. 121, Blackwell Publishing

B. Reflected Wave

For a reflected wave, the travel time is given by the sum of the incident wave time and the reflected wave time, or,

$$t_b = \frac{SC}{c_1} + \frac{CR}{c_1} \tag{13}$$

where $SC = CR = \sqrt{(\frac{1}{2}x)^2 + h^2}$. Thus,

$$t_b = \frac{2}{c_1}\sqrt{h^2 + \frac{x^4}{4}} \tag{14}$$

The graph of $t_b$ is a hyperbola, with $\lim_{x\to\infty} t_b = \frac{x}{c_1}$, i.e. as $x \to \infty$, the travel time curve for a reflected wave approaches the shape of the travel time curve for a direct wave:

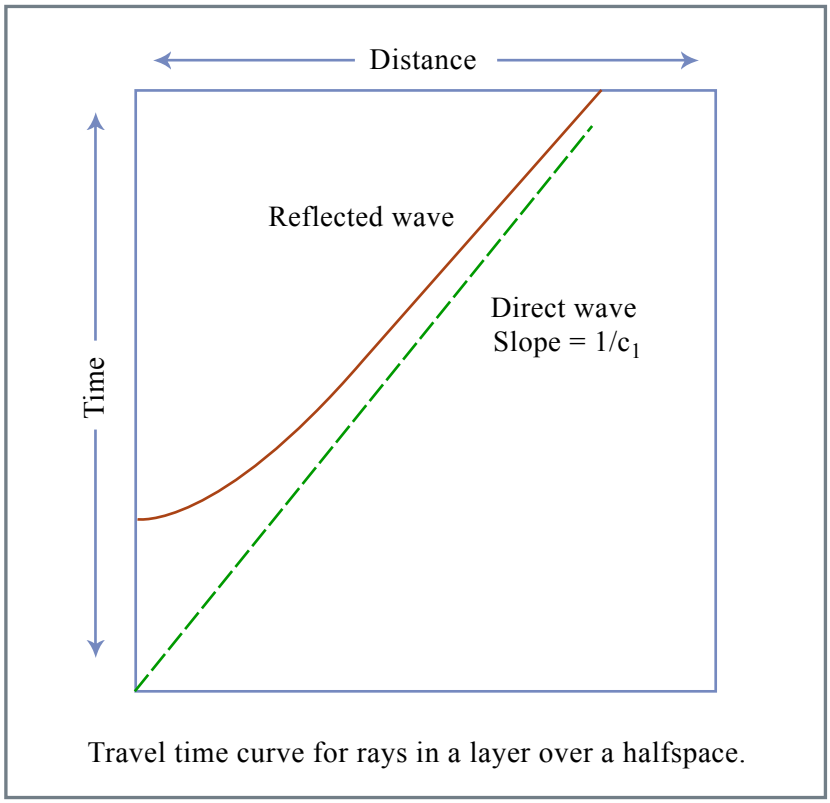



Figure by MIT OCW.

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, fig. 3.2-2, p. 121, Blackwell Publishing

Also note that when $x = 0$, $t_b = \frac{2h}{c_1}$.

C. Head Wave

The travel time for a head wave, which occurs when $i > i_c$, is given by

$$t_c = \frac{SA}{c_1} + \frac{AB}{c_2} + \frac{BR}{c_1} \tag{15}$$

which can be written as,

$$t_c = \frac{x}{c_2} + \frac{2h}{c_1}\sqrt{1 - (\frac{c_1}{c_2})^2} \tag{16}$$

Notice that equation (16) is of the form $t = mx + b$, thus giving a straight line. The $x = 0$ intercept (t-intercept) is then given by,

$$t_c(0) = \frac{2h}{c_1}\sqrt{1 - (\frac{c_1}{c_2})^2} \tag{17}$$

which is less than the value at $x = 0$ for $t_b$.

Now, plotting the three travel time curves together gives the following:

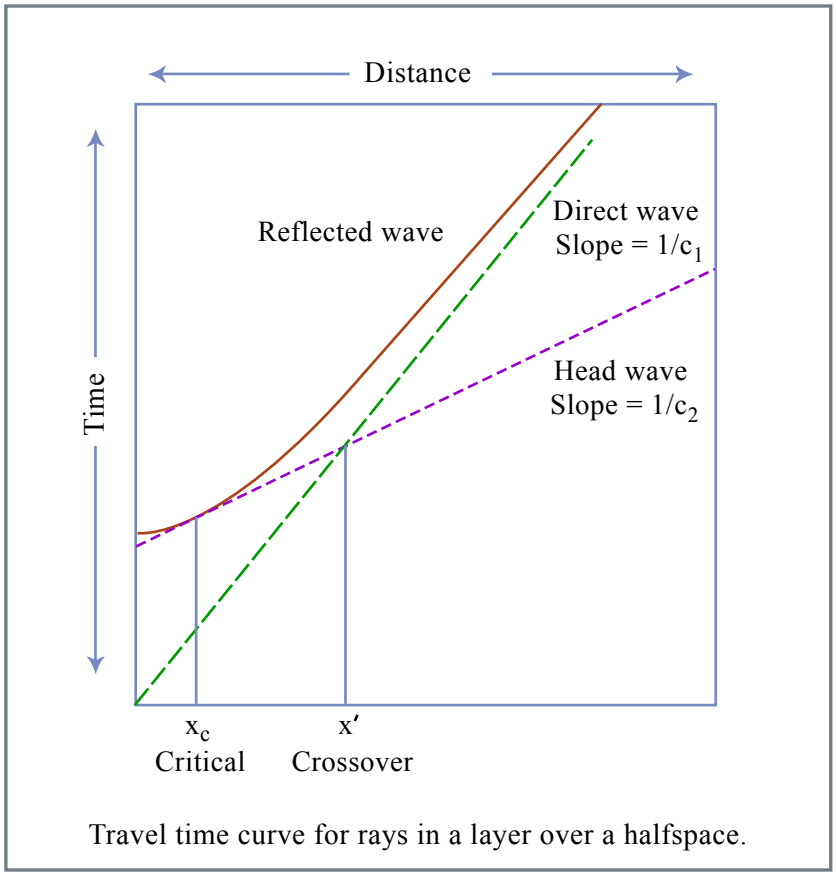



Figure by MIT OCW.

Adapted from Stein & Wysession (2003), *An Introduction to Seismology, Earthquakes, and Earth Structure*, fig. 3.2-2, p. 121, Blackwell Publishing

where $x_c = \frac{2hc_1}{(c_2^2 - c_1^2)^{\frac{1}{2}}}$ and $x' = 2h\sqrt{\frac{c_2+c_1}{c_2-c_1}}$ . Notice that at large values of $x$, the head wave is the first arrival. After $c_1$ and $c_2$ have been determined from the slopes of the travel time curves, the layer thickness $h$ can be found from any of the following:

* t-intercept times, i.e. $t_b(0)$ or $t_c(0)$
* $x_c$ (although it is not very accurate)
* $x'$, which can be inferred easily from observation

Complications on using the travel time curves arise when there is a nonzero focal depth, or when the subsurface geometry is complicated (e.g. a dipping interface or multiple layers).