Surface Waves (Ground Roll)

Today we will look at the interaction of an acoustic wave (ground roll) with a layer over half space. We will consider travel time curves for acoustic waves and describe them using higher modes of surface wave propagation. We will also use a propagation matrix as a reflectivity method to calculate synthetic seismograms.

I. Layer over Source in a Half Space

Let's look at what happens when we have 2 interfaces. We are working with acoustic waves, so we will take P = pressure field. Note that the methods used here can be used for SH waves as well. The same principles can be applied to P-SV waves, but the algebra becomes more complicated.

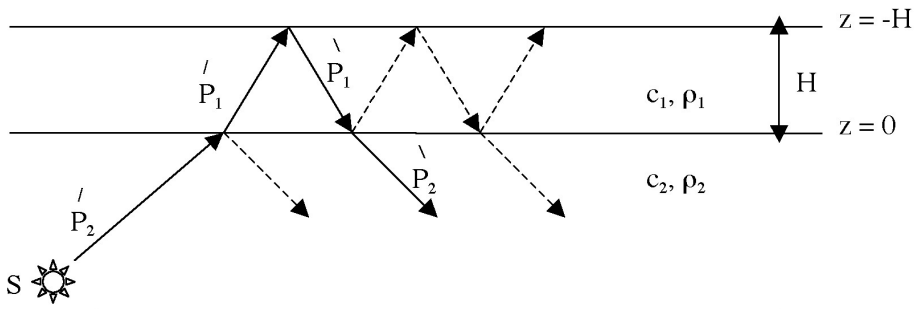


Fig. 1: Diagram of a source in a half space under a layer

The wave equation for the acoustic case is

$$\ddot{P} = k \bigtriangledown \cdot (\frac{1}{p} \bigtriangledown P) \tag{1}$$

with the displacement given by

$$\mathbf{u} = \frac{1}{\rho\omega^2} \bigtriangledown P \tag{2}$$

This case becomes a bit more complicated than the simple reflection previously discussed. The analytical methods used to decribe the simple reflection begin to break down when multiple layers are introduced.

We will define $\overset{/}{P}_2 = 1$ and $\overset{\backslash}{P}_2 = R$. Using the plane wave description,

$$P = \overset{/}{P}\, e^{i(k_x x - k_z z - \omega t)} + \overset{\backslash}{P}\, e^{i(k_x x + k_z z - \omega t)} \tag{3}$$

Notice the positive and negative $k_z z$ terms to describe the vertical slowness, as well as the different amplitudes $\overset{/}{P}$ and $\overset{\backslash}{P}$. We want to know what is happening with respect to crossing the interface, so we will ignore the x-direction, leaving the $e^{ik_z z}$ terms and giving

$$P = \overset{/}{P}\, e^{-i\nu z} + \overset{\backslash}{P}\, e^{i\nu z} \tag{4}$$

where the vertical wave number is given by

$$\nu_z = k_z = \frac{\omega}{c_z} = \frac{\omega cosi}{c} = \omega\eta \tag{5}$$

Equation (4) can be set up for each layer, e.g. $P_1 = \overset{/}{P_1}\, e^{-i\nu_1 z} + \overset{\backslash}{P_1}\, e^{i\nu_1 z}$. We can solve for the pressure by taking the gradient in the z-direction:

$$u_z = \frac{\delta P}{\delta z} = \frac{1}{\rho\omega^2}(i\nu\, \overset{\backslash}{P}\, e^{i\nu z} - i\nu\, \overset{/}{P}\, e^{-i\nu z}) = \frac{i\nu}{\rho\omega^2}(\overset{\backslash}{P}\, e^{i\nu z} - \overset{/}{P}\, e^{-i\nu z}) \tag{6}$$

To solve this system we will follow the steps as before:

1. Look at "potentials" (here, we will look at pressure)
2. Define boundary conditions (e.g. displacement and stress conditions)
3. Set up system of Zoeppritz equations
4. Solve, e.g. for R

At $z = 0$, the stress is continuous, so

$$\overset{/}{P_2} + \overset{\backslash}{P_2} = 1 + R = \overset{/}{P_1} + \overset{\backslash}{P_1} \tag{7}$$

The displacement at $z = 0$ is given by

$$u(0) = \frac{i\nu}{\rho\omega^2}(\overset{\backslash}{P_1} - \overset{/}{P_1}) \tag{8}$$

which implies that

$$\frac{\nu_1}{\rho_1}(\overset{\backslash}{P_1} - \overset{/}{P_1}) = \frac{\nu_2}{\rho_2}(\overset{\backslash}{P_2} - \overset{/}{P_2}) = \frac{\nu_2}{\rho_2}(R - 1) \tag{9}$$

We can also come to the same conclusion using

$$\eta = \frac{\rho_1\nu_2}{\rho_2\nu_1} = \frac{\rho_1 c_1}{\rho_2 c_2} = \frac{z_1}{z_2} \tag{10}$$

which leads to

$$\overset{\backslash}{P_1} - \overset{/}{P_1} = \frac{\nu_2\rho_1}{\nu_1\rho_2}(R - 1) \tag{11}$$

At $z = -H$,

$$\overset{/}{P_1}\, e^{-i\nu_1 H} + \overset{\backslash}{P_1}\, e^{i\nu_1 H} = 0 \tag{12}$$

From here we can write the Zoeppritz equations and solve for R as follows:

$$\begin{bmatrix} 1 & 1 & -1 \\ \frac{-\nu_1}{\rho_1} & \frac{\nu_1}{\rho_1} & \frac{-\nu_2}{\rho_2} \\ e^{-i\nu_1 H} & e^{i\nu_1 H} & 0 \end{bmatrix} \begin{bmatrix} \overset{/}{P}_1 \\ \overset{\backslash}{P}_1 \\ R \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{13}$$

$$R = \frac{(1-\eta) + (1+\eta)e^{2i\nu_1 H}}{(1+\eta) + (1-\eta)e^{2i\nu_1 H}} \tag{14}$$

where $\eta = \frac{\rho_1 \nu_2}{\rho_2 \nu_1}$. Notice that from equation (12),

$$\overset{\backslash}{P}_1 = -e^{2i\eta_1 H} \overset{/}{P}_1 \tag{15}$$

$$\overset{\backslash}{P}_1 = -\overset{/}{P}_1 \tag{16}$$

Some remarks on the reflection at the free surface:

- The reflection coefficient is a complex number. In the cases we studied in previous lectures, R became complex when $i = i_c$.
- $|R| = 1$, which means that the energy is not stored in the upper layer but is eventually all reflected back. E.g., if the input is a single spike, the output is a series of reverberations (see homework 2, problem 3). This occurs because of conservation of energy; if there is no source in the upper layer, there can be no residual trapped energy in the upper layer. The energy is reflected from $z = -H$ and both reflected and partially transmitted from $z = 0$ until it fully dissipates back into the half space.
- There is a frequency-dependent phase shift $\frac{\gamma}{\omega}$ in the waveform, i.e.

$$R = e^{2i\gamma}\ e^{i(kx - \omega t + \frac{\gamma}{\omega})} \tag{17}$$

II. Source Within a Layer Over a Half Space

The source within the layer is analagous to a source in a weathered layer, e.g. an induced source in exploration seismology or surface waves in earthquake seismology. This scenario is similar to the Love waves case discussed in previous lectures. In the case of angles such that $i > i_c$ (evanescent waves), no energy will be transmitted into the half space.

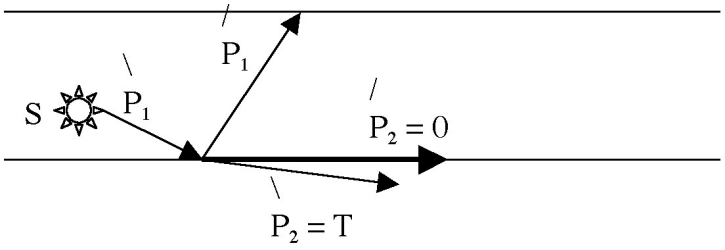


Fig. 2: Diagram of a source in a layer over a half space

We can once again go through the 4 steps to solve for T.

Because stress must be continuous,

$$\overset{\backslash}{P}_1 + \overset{/}{P}_1 = T + 0 = T \tag{18}$$

$$\overset{\backslash}{P}_1 + \overset{/}{P}_1 - T = 0 \tag{19}$$

The total displacement must be continuous, so

$$\mathbf{u} = \frac{1}{\rho\omega^2} \bigtriangledown P = \frac{i\nu}{\rho\omega^2}(\overset{\backslash}{P}_1 \, e^{i\nu z} + \overset{/}{P}_1 \, e^{-i\nu z}) \tag{20}$$

The displacement at $z = 0$ is given by

$$\frac{\nu_1}{\rho_1}(\overset{\backslash}{P}_1 - \overset{/}{P}_1) = \frac{\nu_2}{\rho_2} T \tag{21}$$

$$\overset{\backslash}{P}_1 - \overset{/}{P}_1 = \frac{\rho_1 \nu_2}{\rho_2 \nu_1} T = \eta T \tag{22}$$

At $z = -H$,

$$\overset{\backslash}{P}_1 = -e^{2i\nu_1 H} \, \overset{/}{P}_1 \tag{23}$$

$$\overset{\backslash}{P}_1 + e^{2i\nu_1 H} \, \overset{/}{P}_1 = 0 \tag{24}$$

Notice that the T in these equations is not the conventional T as before because it is evanescent. In contrast to previous cases, we can now form a homogeneous set of equations:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -\eta \\ 1 & e^{2i\nu H} & 0 \end{bmatrix} \begin{bmatrix} \overset{\backslash}{P}_1 \\ \overset{/}{P}_1 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

The solution of this system depends on the layer thickness (H), the vertical wave number (ν), the direction of the rays ($\frac{\omega cosi}{c}$, and the impedence contrast ($\eta = \frac{z_1}{z_2}$. To obtain the nontrivial solution to this system, we must set the determinant of the $3x3$ term to zero, leading to

$$\eta e^{2i\nu_1 H} - e^{2i\nu_1 H} - \eta - 1 = 0 \tag{26}$$

$$(\eta - 1)e^{2i\nu_1 H} = \eta + 1 \tag{27}$$

$$e^{2i\nu_1 H} = \frac{\eta + 1}{\eta - 1} \tag{28}$$

We know that in the complex plane, generally, $e^{-2i\theta} = \frac{a-b}{a+b}$, giving $\tan\theta = -\frac{ib}{a}$. Using this, we can rewrite our nontrivial solution as

$$\tan(\nu_1 H) = \frac{1}{i\eta} = \frac{\rho_2 \nu_1}{i\rho_1 \nu_2} \tag{29}$$

where $\nu_1 = \omega\eta = \omega\sqrt{\frac{1}{c_1^2} - p^2}$. The dispersion relationship for ground roll (which is the same as the dispersion relationship for Love waves) is thus given by

$$\tan\left(\omega H\sqrt{\frac{1}{c_1^2} - p^2}\right) = \frac{\rho_2\sqrt{\frac{1}{c_1^2} - p^2}}{\rho_1\sqrt{p^2 - \frac{1}{c_2^2}}} \tag{30}$$

Let's look at this solution graphically. Setting each side of equation (30) to zero, we obtain the case for (1) a direct wave, or $p = \frac{1}{c_1}$. We also know that for (2) a critical wave, $p = \frac{1}{c_2} = \frac{\sin i_c}{c_1}$.

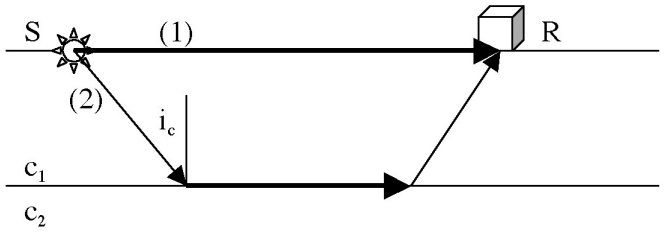


Fig. 3: Diagram of (1) a direct wave and (2) a critical wave

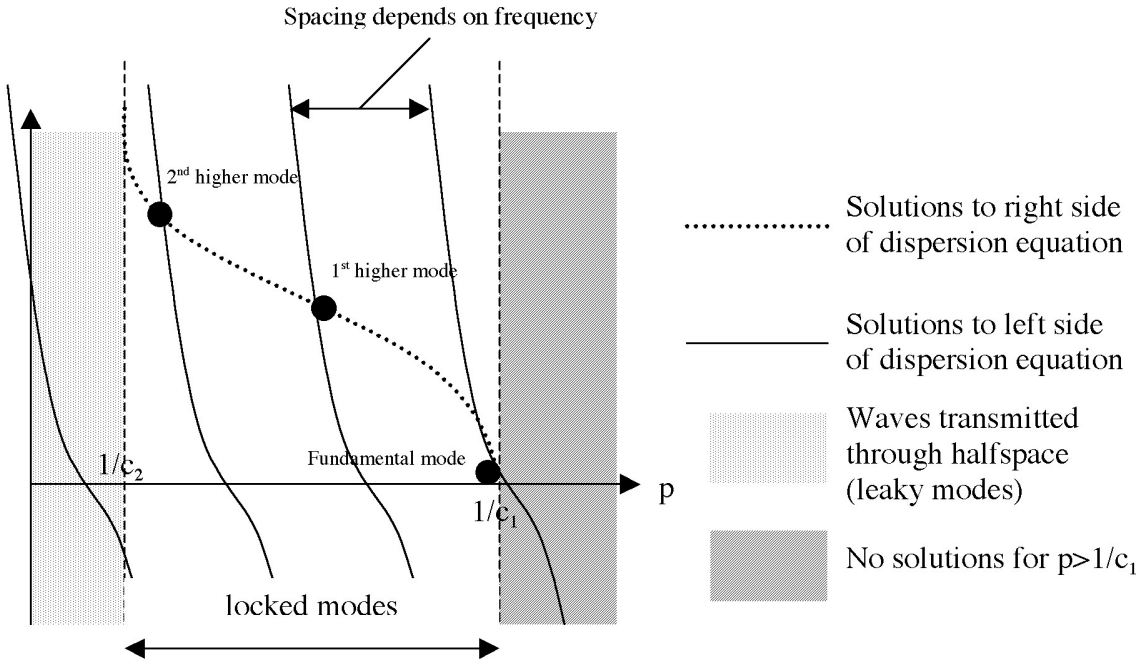


Fig. 4: Graphical representation of the dispersion relationship for ground roll

For all postcritical angles i, or $\frac{1}{c_2} < p < \frac{1}{c_1}$, we have "locked" modes, where all of the energy stays in the upper layer. The fundamental mode labeled at $\frac{1}{c_1}$ represents the p-value for a direct wave. The subsequent intersections of the left side solutions and the right side solutions give progressively higher modes (overtones) as p decreases to $\frac{1}{c_2}$. For precritical angles i, or $p < \frac{1}{c_2}$, energy is lost due to wave transmission into the half space, so we get "leaky" modes. Also, because the spacing between the asymptotes of the left side solutions is dependent on ω, for any given frequency, there is a finite set of solutions (i.e. modes) to the dispersion relation. Each mode has a specific horizontal slowness and take-off angle because the waves can only propagate in a way that creates constructive interference.

As the frequency decreases and wavelength increases, the spacing between the left side solution curves increases, thus decreasing the number of overtones. Similarly, as the frequency increases, the number of overtones also increases. A plot of these overtones as frequency approaches infinity will approach a continuous graph along the plot of the right side solution.

Another way to think about overtones is to visualize the direct wave on Figure 3 with a wavelength equal to the distance between the source and the receiver. This is a high wavelength with a low frequency, and in this case a plot of the dispersion relation would show that the left side solutions are spaced so far apart that few, if any, overtones exist, i.e. given the particular long wavelength and H-value, there are few other incident angles that will result in a coincidence between the path of the wave and the receiver (constructive interference). As the frequency increases and wavelength decreases, there will be more incident angles such that the path of the wave coincides with the receiver and results in constructive interference.