Introduction to Seismology: Lecture Notes
28 February 2005

SNELL'S LAW REVISITED

General approach
1. Potentials
2. BCs
3. Solve the system of equations
4. Reflection and transmission coefficients

Recall Helmholtz representation: $\mathbf{u} = \nabla\phi + \nabla\times\boldsymbol{\psi}$, $\nabla\cdot\boldsymbol{\psi} = 0$,
SV → solve for ϕ and $\boldsymbol{\psi}$ → reconstruct **u**
SH → u_y satisfies wave equation.

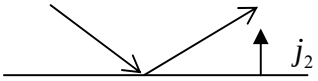
Therefore, SH part doesn't require potentials to find solutions. But you could analyze the SH part through potentials also by using the following Helmholtz presentation of the shear wave potential:
$\boldsymbol{\psi} = \nabla\psi_{SV} + \nabla\times\boldsymbol{\psi}_{SH}$, $\nabla\cdot\boldsymbol{\psi}_{SH} = 0$
so

$$\mathbf{u} = \underbrace{\nabla\phi}_{P} + \underbrace{\nabla\times\nabla\psi_{SV}}_{SV} + \underbrace{\nabla\times\nabla\times\boldsymbol{\psi}_{SH}}_{SH} \quad , \nabla\cdot\boldsymbol{\psi}_{SH} = 0$$

See Stein and Wysession for a good development.

But, customarily, we do not solve the problem this expression.

In general, P-SV and SH waves are *not* decoupled. They are only if the medium is isotropic (heterogeneity is permissible) and if the normal, $\hat{\mathbf{n}}$, at an interface is in the plane of propagation.

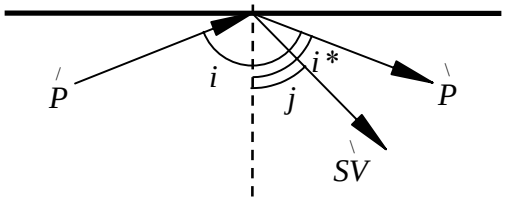


Therefore, if the interface dips left or right with respect to the page, the P-SV and SH systems are still decoupled, but if the interface dips transverse to the page then the P-SV and SH systems are no longer decoupled - $\hat{\mathbf{n}}$ no longer lies within the plane of propagation.

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Another way to approach Snell's Law … consider:

Figure 1



Free surface BC requires that $T_z = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz}) = \mathbf{0}$, and the stress components contain expressions of the form:

$$\omega\left(\frac{\sin i}{\alpha}x - t\right), \omega\left(\frac{\sin i*}{\alpha}x - t\right), \omega\left(\frac{\sin j}{\beta}x - t\right).$$

As the stress components vanish, they must all be equal, hence:

$$\frac{\sin i}{\alpha} = \frac{\sin i*}{\alpha} = \frac{\sin j}{\beta} \equiv p, \text{ ray parameter } \rightarrow i = i*.$$

Now consider the interface between two layers of differing elastic properties …

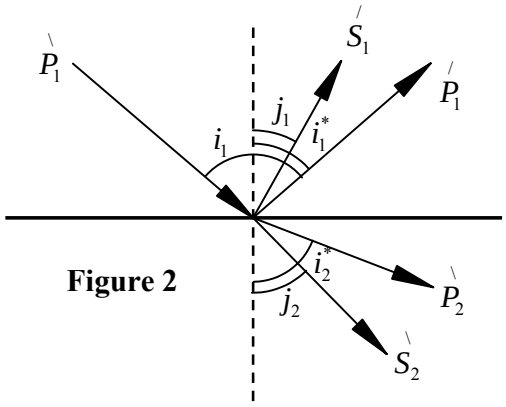


Figure 2

As before, @ z = 0, the displacement and traction elements contain the following expressions:

$$\left(\frac{\sin i_1}{\alpha_1} - t\right), \left(\frac{\sin i_1^*}{\alpha_1} - t\right), \left(\frac{\sin j_1}{\beta_1} - t\right), \left(\frac{\sin i_2^*}{\alpha_2} - t\right), \left(\frac{\sin j_1}{\beta_2} - t\right),$$

all of which must be equivalent to satisfy the kinematic and dynamic boundary conditions. Hence,

$$\frac{\sin i_1}{\alpha_1} = \frac{\sin i_1^*}{\alpha_1} = \frac{\sin i_2^*}{\alpha_2} = \frac{\sin j_1}{\beta_1} = \frac{\sin j_2}{\beta_2} \equiv p$$

The ray parameter is constant for the entire wave field!

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From a geometric point of view, consider plane wave propagation across several interfaces, as below. The wave vector is different in each layer, but the horizontal distance (Δx) traveled along any interface (over an interval of time, Δt) must be the same between interfaces, because the BCs require that wave fronts must be continuous across boundaries. Hence, the apparent horizontal velocity (c_x) is everywhere constant. And since c_x is constant, it follows that $p = 1/c_x$ (the ray parameter) is constant for the entire wave field.

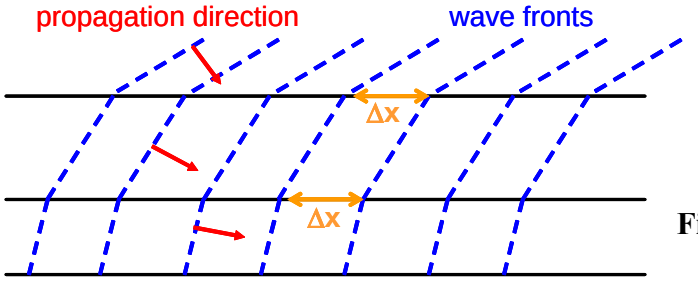


Figure 3

This concept can be extended to a spherical body, but you will find that a singularity exists at the point where a downgoing raypath becomes horizontal, as it inflects from its downgoing to its upgoing branch.

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Normally, $\alpha = \left(\frac{\kappa + 4/3\,\mu}{\rho}\right)^{1/2} > \beta = \left(\frac{\mu}{\rho}\right)^{1/2} \rightarrow 1/\alpha < 1/\beta$. And since $\frac{\sin i}{\alpha} = \frac{\sin j}{\beta}$, it follows that $j < i$. This means that converted or reflected S-waves will usually travel raypaths bent more toward the normal than transmitted or reflected P-waves. Another way of seeing this is by the Dispersion Relation. The dispersion relation states that $k = \left(k_x^2 + k_z^2\right) = \frac{\omega}{c}$. From this fact, the shear and compressional components of the wave vector are given by:

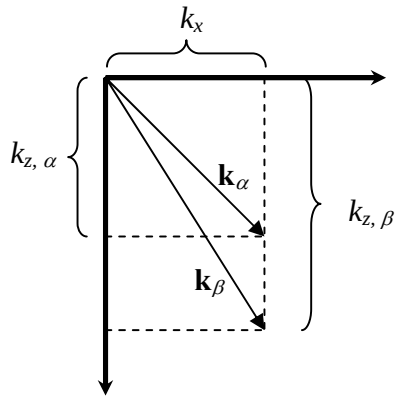


Figure 4

$$\text{S-wave: } \frac{\omega}{\beta}\hat{\mathbf{k}} = \mathbf{k}_\beta$$

$$\text{P-wave: } \frac{\omega}{\alpha}\hat{\mathbf{k}} = \mathbf{k}_\alpha$$

But $\beta < \alpha$, so $\|\mathbf{k}_\beta\| > \|\mathbf{k}_\alpha\|$, and since k_x is a constant, it follows that $k_{z,\beta} > k_{z,\,\alpha}$, meaning that the S-wave propagation direction is typically nearer to the normal than is the P-wave propagation direction.

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CRITICAL ANGLES

1. Free Surface (P-SV system)

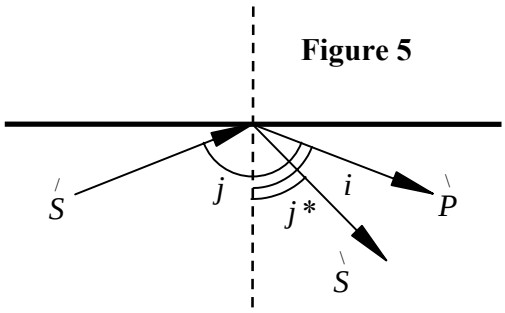


Figure 5

The incident angle at which an incoming S-wave is first critically reflected is the angle j_c that causes i = 90º.

$$\frac{\sin j_c}{\beta} = \frac{\sin i}{\alpha} = \frac{1}{\alpha} \rightarrow j_c = \sin^{-1}\left(\frac{\beta}{\alpha}\right)$$

The converted P-wave is critically reflected, meaning that $k_{z,\alpha} = 0$. So, $\mathbf{s} = (p, \eta_\alpha) \xrightarrow[\text{critical reflection}]{} (p, 0)$.

Recall that $p^2 + \eta_\alpha^2 = \frac{1}{\alpha^2}$. But $\eta_\alpha = 0 \rightarrow p^2 = 1/\alpha^2$ or

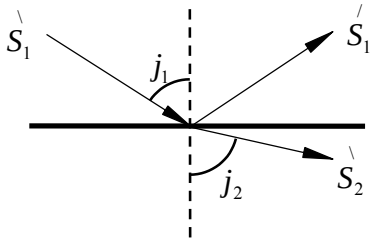
$$p = \frac{\sin j_c}{\beta} = \frac{1}{\alpha}\text{; The ray parameter is exactly } 1/\alpha \text{ at the point of critical reflection}$$

Sub-critical: $j < j_c \leftrightarrow \eta_\alpha > 0$

Critical: $j = j_c \leftrightarrow \eta_\alpha = 0$

Post-critical: $j > j_c \leftrightarrow \eta_\alpha$ is complex (evanescent waves!)

2. Welded interface (SH case)



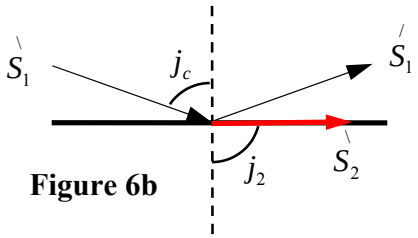


Figure 6b

$$\beta_1 > \beta_2 \rightarrow j_2 > j_1$$

$$@\, j_c,\ j_2 \rightarrow 90^\circ$$

3. Welded interface (SV case)

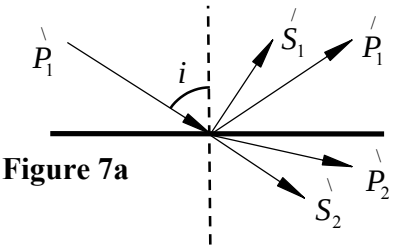


Figure 7a

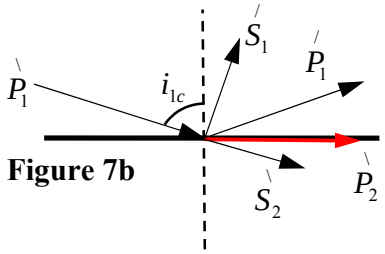


Figure 7b

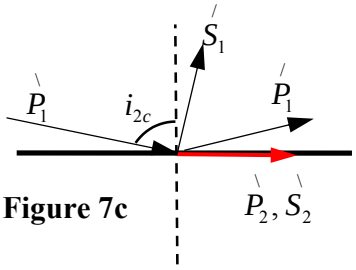


Figure 7c

Figure 7a: Reference case
Figure 7b: 1st critical reflection; i_{1c} is such that downgoing P_2 is critically reflected at the interface.
Figure 7c: 2nd critical reflection; i_{2c} is such that downgoing S_2 is also critically reflected at the interface. Note that downgoing P_2 is now post-critical. $i_{2c} > i_{1c}$.

Note that the transmission and refection coefficients will be real-valued for sub-critical phenomena but become complex-valued for post-critical.

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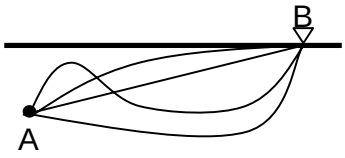
FERMAT'S PRINCIPLE

Hamiltonian variational principle · "Least action"

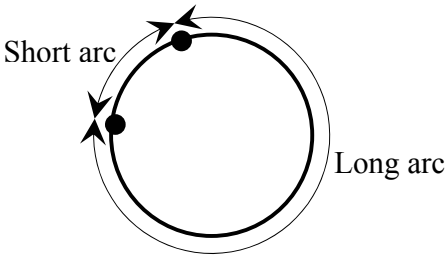
Consider the kinetic and potential energy along an arbitrary path between two points, A and B. Stationary points of the integrated difference between KE and PE over all possible paths specify paths of "least action".

Optical domain

There are an infinite number of paths from A to B, but there is only one 'correct' path: the one with the shortest travel time.



Both the shortest and the longest paths are stationary points.



Casting the problem mathematically …

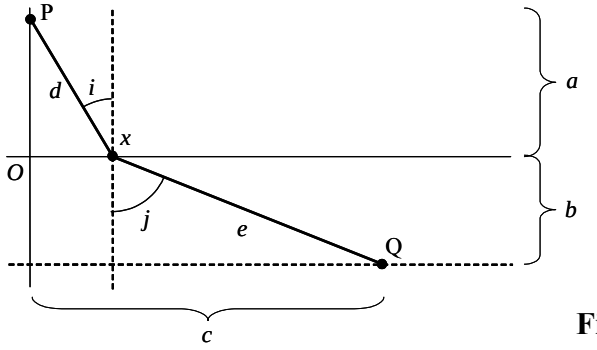


Figure 8

$$T_{P-Q} = \frac{d}{c_1} + \frac{e}{c_2}$$

$$= \frac{\left(a^2 + x^2\right)^{1/2}}{c_1} + \frac{\left(b^2 + (c-x)^2\right)^{1/2}}{c_2}$$

$$\frac{dT}{dx} = \underbrace{\frac{x}{c_1\left(a^2 + x^2\right)^{1/2}}}_{} + \underbrace{\frac{c-x}{c_2\left(b^2 + (c-x)^2\right)^{1/2}}}_{} = \underbrace{\frac{\sin i}{c_1}}_{} - \underbrace{\frac{\sin j}{c_2}}_{} = 0 \quad (optimum!)$$

Hence, $\frac{\sin i}{c_1} = \frac{\sin j}{c_2}$ (Snell's Law). So Fermat's Principle implies Snell's Law.

■

The travel time curve, plotted as a function of offset, is typically a hyperbolic function. Near stationary points, the curve is usually fairly 'flat', which implies that, near optimum or stationary points, the travel time is locally insensitive to slight variations in offset. Consequently, close to a stationary point, small deviations in the ray path can be treated as second order effects.

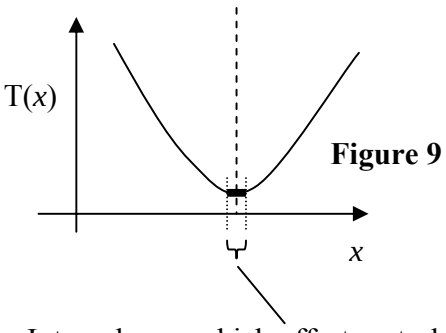
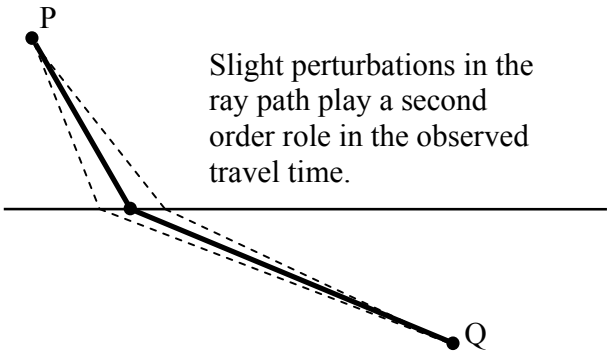


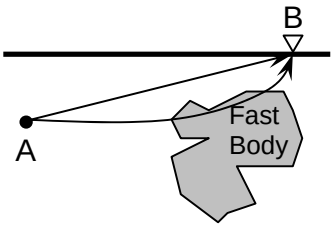
Figure 9

Interval over which offset perturbations can be treated as secondary effects.



Slight perturbations in the ray path play a second order role in the observed travel time.

For example …



- The presence of a fast body embedded in a homogeneous medium causes the reference (optimal) ray path to deflect from its original path between A and B.
- But, as a consequence of Fermat's Principle, the change in ray path produces a second order effect on the arrival time.
- The effect on travel time of changes in *wave speed* (elastic parameters) is first order.

Implication: We can generate reference ray paths assuming a homogeneous medium and use this reference model to simplify subsequent analysis for heterogeneous media. → Important for travel time tomography.

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REFLECTION AND TRANSMISSION COEFFICIENTS

Reflection coefficient: $R \equiv \frac{A_{\text{reflected}}}{A_{\text{incoming}}}$

Transmission coefficient: $T \equiv \frac{A_{\text{transmitted}}}{A_{\text{incoming}}}$

SH Case

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Consider an SH-wave incident at an interface between two layers of different elastic properties. Because SH-waves are decoupled from P and SV waves, the transmitted and reflected waves will also be SH waves.
The displacement of the downgoing wave in the Layer 1 (see Figure 10) can be expressed as

$$\overset{\backslash}{u}_1(\mathbf{x},t) = \overset{\backslash}{A}_1\, g(p,\eta_1,\mathbf{x},t)$$

where, for notational convenience

$$g(p,\eta_1,\mathbf{x},t) \equiv \exp[i\omega(px+\eta_1 z - t)]$$
$$\eta_1 \equiv \eta_{1,\beta}$$
$$\mathbf{x} \equiv (x,z)$$

and the reflected, upgoing wave in Layer 1 can be expressed as

$$\overset{/}{u}_1(\mathbf{x},t) = \overset{/}{A}_1\, g(p,-\eta_1,\mathbf{x},t).$$

Now, setting the interface to $z = 0$, the total displacement in Layer 1 is

$$\overset{/}{u}_1 + \overset{\backslash}{u}_1 = \overset{/}{A}_1\, g(p,\eta_1,\mathbf{x},t) + \overset{\backslash}{A}_1\, g(p,-\eta_1,\mathbf{x},t).$$

The displacement in Layer 2 is given by

$$\overset{\backslash}{u}_2 = \overset{\backslash}{A}_2\, g(p,\eta_2,\mathbf{x},t).$$

The kinematic boundary condition requires that the displacement be continuous across the interface at $z = 0$. Therefore, setting $z = 0$ and equating the displacements in Layers 1 and 2, we arrive at the following statement:

$$\overset{/}{u}_1 + \overset{\backslash}{u}_1 = \overset{\backslash}{u}_2 \Leftrightarrow \overset{/}{A}_1 + \overset{\backslash}{A}_1 = \overset{\backslash}{A}_2 \quad (1)$$

and

$$R = \frac{\overset{/}{A}_1}{\overset{\backslash}{A}_1}$$

$$T = \frac{\overset{\backslash}{A}_2}{\overset{\backslash}{A}_1}$$

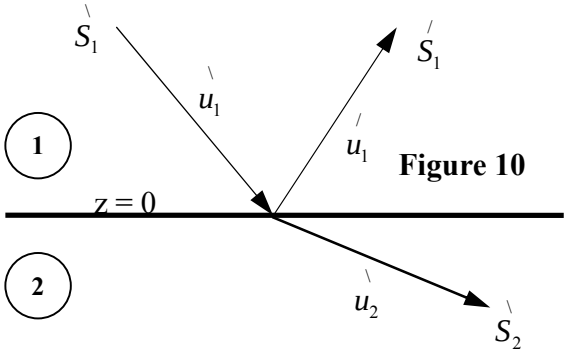


Figure 10

The dynamic boundary conditions require that

$$\sigma_{ij} = \lambda\delta_{ij}\Delta + 2\mu\varepsilon_{ij} = \lambda\delta_{ij}\Delta + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and recalling , for SH waves, $\mathbf{u} = (0, u_y, 0)$, we have:

$$\sigma_{xz} = 0,\ \ \sigma_{yz} = \mu\frac{\partial u_y}{\partial z},\ \ \sigma_{zz} = 0$$

$$\overset{\backslash}{\sigma}_{yz,1} = \mu_1 \frac{\partial \overset{\backslash}{u}_{y,1}}{\partial z} = i\omega\mu_1\eta_1 \overset{\backslash}{A}_1\, g(p, \eta_1, \mathbf{x}, t)$$

$$\overset{/}{\sigma}_{yz,1} = \mu_1 \frac{\partial \overset{/}{u}_{y,1}}{\partial z} = -i\omega\mu_1\eta_1 \overset{/}{A}_1\, g(p, -\eta_1, \mathbf{x}, t)$$

$$\overset{\backslash}{\sigma}_{yz,2} = \mu_1 \frac{\partial \overset{\backslash}{u}_{y,2}}{\partial z} = i\omega\mu_2\eta_2 \overset{\backslash}{A}_2\, g(p, \eta_2, \mathbf{x}, t)$$

hence, @ $z = 0$,

$$\sigma_1 = \overset{/}{\sigma}_1 + \overset{\backslash}{\sigma}_1 = i\omega\mu_1\eta_1\left(\overset{\backslash}{A}_1 - \overset{/}{A}_1\right)\exp[i\omega(px - t))]$$

$$\sigma_2 = \overset{\backslash}{\sigma}_2 = i\omega\mu_2\eta_2 \overset{\backslash}{A}_2 \exp[i\omega(px - t))]$$

The preceding facts imply that

$$\mu_1\eta_1\left(\overset{\backslash}{A}_1 - \overset{/}{A}_1\right) = \mu_2\eta_2 \overset{\backslash}{A}_2 \quad \ldots(2)$$

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The reflection and transmission coefficients are expressed as

$$R_{ss} = \overset{\backslash}{S}\,\overset{/}{S} = \frac{\overset{/}{A}_1}{\overset{\backslash}{A}_1}$$

$$T_{ss} = \overset{\backslash}{S}\,\overset{\backslash}{S} = \frac{\overset{\backslash}{A}_2}{\overset{\backslash}{A}_1}$$

■

If we re-arrange Equ. 2.

$$\overset{/}{A}_1 - \overset{\backslash}{A}_1 = -\overset{\backslash}{A}_2 \frac{\mu_2\eta_2}{\mu_1\eta_1}$$

Combining equs. 1 and 2 give the **Zoeppritz Equation**

$$\overset{\backslash}{A}_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 1 & \dfrac{\mu_2\eta_2}{\mu_1\eta_1}\end{pmatrix}\begin{pmatrix}\overset{/}{A}_1\\ \overset{\backslash}{A}_2\end{pmatrix}$$

Where the left hand side (LHS) is the incoming wave and the RHS are the resulting waves.

Again, if we re-arrange the Zoeppritz Equ:

$$\left(\overset{/}{A}_1-\overset{\backslash}{A}_1\right)\mu_1\eta_1=-\left(\overset{\backslash}{A}_1+\overset{/}{A}_1\right)\mu_2\eta_2$$

$$\overset{/}{A}_1\left(\mu_1\eta_1+\mu_2\eta_2\right)=\overset{\backslash}{A}_1\left(\mu_1\eta_1-\mu_2\mu_2\right)\text{s}$$

Then the **reflection coefficient** is:

$$R\equiv\frac{\overset{/}{A}_1}{\overset{\backslash}{A}_1}=\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}$$

And to calculate the **transmission coefficient**:

$$\overset{\backslash}{A}_2=\overset{\backslash}{A}_1+\overset{/}{A}_1=\overset{\backslash}{A}\left(1+\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}\right)=\overset{\backslash}{A}_1\left(\frac{2\mu_1\eta_1}{\mu_1\eta_1+\mu_2\eta_2}\right)$$

$$T=\frac{\overset{\backslash}{A}_2}{\overset{\backslash}{A}_1}=\frac{2\mu_1\eta_1}{\mu_1\eta_1+\mu_2\eta_2}$$

The dependence on the incidence angle, i_1, is illustrated by

$$\eta_1=\frac{\cos i_1}{\beta_1},\eta_2=\frac{\cos i_2}{\beta_2}$$

And

$$\beta^2=\frac{\mu}{\rho}$$

Using these variables to re-write the reflection and transmission coefficients:

$$R=\frac{\rho_1\beta_1\cos i_1-\rho_2\beta_2\cos i_2}{\rho_1\beta_1\cos i_1+\rho_2\beta_2\cos i_2}=\frac{\rho_1\beta_1-\rho_2\beta_2}{\rho_1\beta_1+\rho_2\beta_2}$$

$$T=\frac{2\rho_1\beta_1\cos i_1}{\rho_1\beta_1\cos i_1+\rho_2\beta_2\cos i_2}=\frac{2\rho_1\beta_1}{\rho_1\beta_1+\rho_2\beta_2}$$

The reflection and transmission coefficients are equal to the RHS of the equations above for the specific case of normal incidence, when i_1=i_2=0.

For a welded interface we can also have $\overset{/}{A}_2 \rightarrow \overset{/}{A}_1$ *and* $\overset{\backslash}{A}_2$

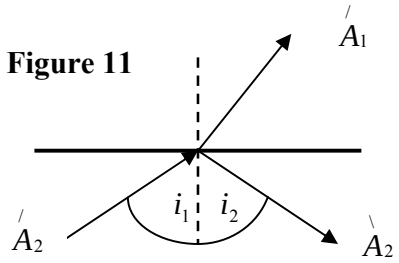


Figure 11

Thus we can define a scatter matrix S:

$$\underline{\underline{S}} = \begin{pmatrix} \overset{/}{S}\overset{/}{S} & \overset{\backslash}{S}\overset{/}{S} \\ \overset{/}{S}\overset{\backslash}{S} & \overset{\backslash}{S}\overset{\backslash}{S} \end{pmatrix}$$

(Rob showed figure on overhead of R and T versus angle of incidence for pre-critical and critical angles. The reflection coefficient lies below 0 for i_1=0. At the intramission angle the reflection coefficient is 0 and increases to a value of 1 and the critical angle, i_c. The transmission coefficient is less than 1 at i_1=0 and increases to a value of 2 at the critical angle, i_c.)