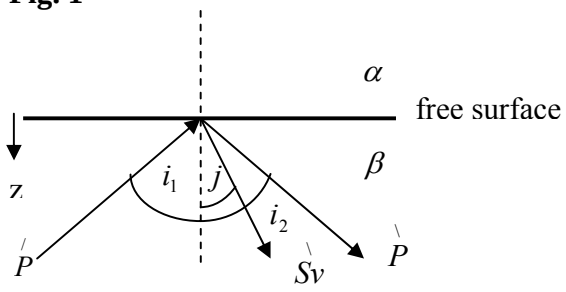
Introduction to Seismology:Lecture Notes
2nd March 2005

P-Sv CASE

A P wave incident on a free surface generates both reflected P and Sv waves (Fig. 1)

Fig. 1



$$\frac{\sin i_1}{\alpha} = \frac{\sin j}{\beta} = \frac{\sin i_2}{\beta}, \alpha \rangle \beta \Rightarrow j \langle i_1$$

We follow the same approach as the S_H case:

1. Potentials
2. Boundary conditions
3. Solve system of equations (often in the matrix form of Zoeppritz's Equations)

Potentials

$$\phi_I = A \exp\{i\omega(px - \eta_\alpha z - t)\}$$

$$\phi_R = B \exp\{i\omega(px + \eta_\alpha z - t)\} \quad \text{P wave}$$

$$\varphi_R = C \exp\{i\omega(px + \eta_\beta z - t)\} \quad \text{Sv wave}$$

Boundary Conditions

The traction on the free surface must vanish

$$T_3 = 0 = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz}) = 0$$

As before

$$\sigma_{ij} = \lambda \delta_{ij} \Delta + 2\mu \varepsilon_{ij}$$

$$\sigma_{zz} = \lambda \nabla^2 \phi + 2\mu \frac{\partial^2 \phi}{\partial z^2} + 2\mu \frac{\partial^2 \varphi}{\partial x \partial z}$$

$$\sigma_{xz} = 2\mu \frac{\partial^2 \phi}{\partial x \partial z} + \mu \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial z^2} \right)$$

Then solve

$$\overset{/}{A}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\quad&\quad\end{pmatrix}\begin{pmatrix}\overset{\backslash}{B}\\\overset{\backslash}{C}\end{pmatrix}$$

Which gives

$$(A+B)\left\{(\lambda+2\mu)\eta_\alpha{}^2+p^2\lambda\right\}+C(2\mu p\eta_\beta)=0$$
$$(A-B)2p\eta_\alpha-C\left(p^2-\eta_\beta^2\right)=0$$

After a bit of work we can get expressions of the reflection coefficient (R_{pp}) and the conversion/refraction coefficient (R_{ps}).

$$R_{\overset{/}{p}\overset{\backslash}{p}}=\overset{/}{P}\overset{\backslash}{P}=\frac{\overset{\backslash}{B}}{\overset{/}{A}}=\frac{(\lambda+2\mu)\eta_\alpha^2+p^2\lambda+4\mu p^2\eta_\alpha\eta_\beta\left(p^2-\eta_\beta^2\right)^{-1}}{-\left\{(\lambda+2\mu)\eta_\alpha^2+p^2\lambda\right\}+4\mu p^2\eta_\alpha\eta_\beta\left(p^2-\eta_\beta^2\right)^{-1}}$$

$$R_{\overset{/}{P}\overset{\backslash}{S}}=\overset{/}{P}\overset{\backslash}{S}=\frac{\overset{\backslash}{C}}{\overset{/}{A}}=\left(\frac{4p\eta_\alpha}{p^2-\eta_\beta^2}\right)\left\{\frac{(\lambda+2\mu)\eta_\alpha^2+p^2\lambda}{(\lambda+2\mu)\eta_\alpha^2+p^2\lambda-4\mu p^2\eta_\alpha\eta_\beta(p^2-\eta_\beta^2)^{-1}}\right\}$$

(See figure on overhead of P and Sv reflection coefficients versus angle of incidence)

There are several things that should be noted about these equations:

1. The coefficients are dependent on the incidence angle, through
   $\eta_\alpha=\frac{\cos i_1}{\alpha},\eta_\beta=\frac{\cos j}{\beta}$ and $p=\frac{\sin i_1}{\alpha}$
2. For normal incidence the case is more simple
   $i_1=0\rightarrow p=0$
   $\eta_\alpha=\frac{1}{\alpha};\eta_\beta=\frac{1}{\beta}$
   so
   $R_{pp}=-1$, a full reflection with a change in polarity
   $R_{PS}=0$, there is no P$\rightarrow$Sv conversion
   $R_{SP}=\overset{/}{S}\overset{\backslash}{P}=0$
   $R_{SS}=\left|\overset{/}{S}\overset{\backslash}{S}\right|=1$
3. Notice that R_{PP} and R_{SS} do not depend on frequency, ω. This is only true so long as :

$\frac{1}{c^2} - p^2 > 0$ so that $\eta = \sqrt{\frac{1}{c^2} - p^2}$ is a real number.

We can now do the same for the incoming Sv wave R_{SP} and repeat the exercise for waves coming from above (Fig. 2).

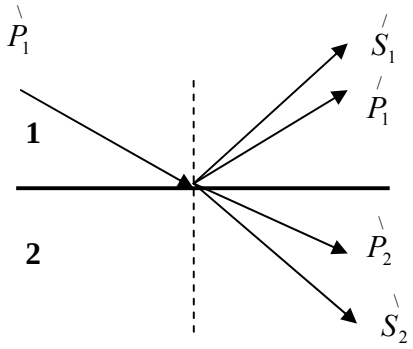


Figure 2

welded interface

This gives the scatter matrix (or "dispersion" matrix)

$$S = \begin{pmatrix} PP & SP & SP & PP \\ PS & SS & SS & PS \\ PP & SP & SP & PP \\ PS & SS & SS & PS \end{pmatrix}$$

4. Poisson Medium, when $\lambda = \mu$, leads to Rayleigh wave system (more later in course).

CRITCAL AND POST-CRITICAL REFLECTION

Figure 3 shows the pre-critical situation.

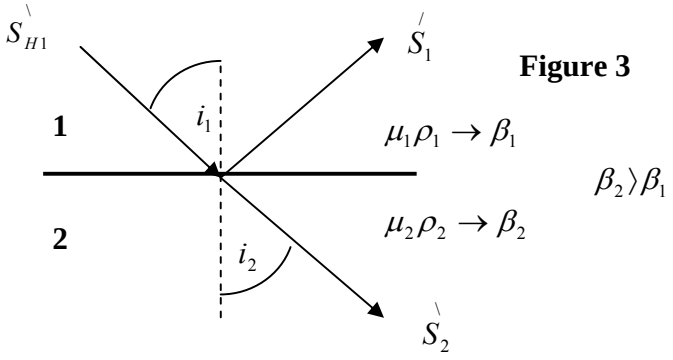


Figure 3

$$R_{SS'} = \frac{\rho_1\beta_1 \cos i_1 - \rho_2\beta_2 \cos i_2}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2}$$

$$T_{SS} = \frac{2\rho_1\beta_1 \cos i_1}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2}$$

If we increase i_1 to the critical angle, i_c (Fig. 4).

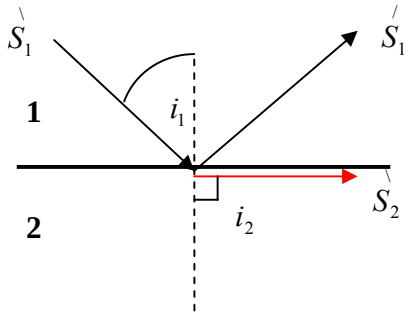


Figure 4

$$i_1 = i_c, i_2 = \frac{\pi}{2}$$

$$p = \frac{\sin i_1}{\beta_1} = \frac{\sin i_c}{\beta_1} = \frac{\sin i_2}{\beta_2} = \frac{1}{\beta_2} \Rightarrow \sin i_c = \frac{\beta_1}{\beta_2}$$

$\eta_\beta = \sqrt{\frac{1}{(\beta_2)^2} - \frac{1}{(\beta_2)^2}} = 0$ therefore there is no vertical propagation, k_z=0

$R_{SS'} = S\hat{S}S' = \frac{\rho_1\beta_1 \cos i_1 - \rho_2\beta_2 \cos i_2}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2} = 1$, a full reflection

$T_{SS} = S\hat{S}S = \frac{2\rho_1\beta_1 \cos i_1}{\rho_1\beta_1 \cos i_1 + \rho_2\beta_2 \cos i_2} = 2$ (also known as the Head Wave)

When the incidence angle is increased to post-critical, i.e. $i_1 \rangle i_c$, i_2 cannot increase to an angle greater than 90º.

$$\frac{\sin i_1}{\beta_1} \rangle \frac{\sin i_c}{\beta_1} = p = \frac{1}{\beta_2}$$

For the incoming wave $p \rangle \frac{1}{\beta_2}$

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$$\eta_2 = \sqrt{\frac{1}{(\beta_2)^2} - p^2} = i\sqrt{p^2 - \frac{1}{(\beta_2)^2}} = i\hat{\eta}_2$$

η_2 is now a complex number

For an SH wave the displacement vector is **U**= (0,U_y,0)

$$U_{y,2} = \exp\{i\omega(px + \eta_2 z - t)\}$$

In a postcritical situation when $\eta_2 = i\hat{\eta}$

$$U_{y,2} = \exp\{iw(px + i\hat{\eta}_2 z - t)\}$$

$$U_{y,2} = \exp\{-\omega\hat{\eta}_2 z\}\exp\{i\omega(px - t)\}$$

Where the first term on the RHS describes an exponential decay in the z- direction (there is no propagation in the z-direction). The frequency, ω, controls the rate of decay. The second term on the RHS describes a harmonic function (**x**,t), therefore the wave propagation is in the *x*-direction.

$$U_{y,2} = A(z)\exp\{i\omega(px - t)\}$$

The property of decreasing wave amplitude with depth based on the frequency of the wave is known as **Evanescence.**

$$\omega\eta = k_z = \frac{2\Pi}{\lambda}$$

$$\eta = \frac{2\Pi}{\lambda\omega}$$

A wave with a short wavelength (λ), high frequency (ω), will decay more quickly. At infinite frequency the decay is instantaneous and the wave becomes a ray.

If the properties of a medium change with depth, for example there is a body which allows a wave to pass through it more quickly at depth only the low ω, long λ, waves will sample it as the high ω waves will have been stripped out. A wave with a frequency dependence is a **Dispersive wave.** Often a wave can be dispersive and evanescent.

(See figure on overhead of post critical reflection and transmission coefficients)