Localization of the Complex Spectrum: The S Transform

R. G. Stockwell, L. Mansinha, and R. P. Lowe

Abstract

The S transform, which is introduced in this correspondence, is an extension of the ideas of the continuous wavelet transform (CWT), and is based on a moving and scalable localizing Gaussian window. It is shown here to have some desirable characteristics that are absent in the continuous wavelet transform. The S transform is unique in that it provides frequency-dependent resolution while maintaining a direct relationship with the Fourier spectrum. These advantages of the S transform are due to the fact that the modulating sinusoids are £xed with respect to the time axis, whereas the localizing scalable Gaussian window dilates and translates.

I. Introduction

In geophysical data analysis and in many other disciplines, the concept of a stationary time series is a mathematical idealization that is never realized and is not particularly useful in the detection of signal arrivals. Although the Fourier transform of the entire time series does contain information about the spectral components in a time series, for a large class of practical applications, this information is inadequate. An example from seismology is an earthquake seismogram. The £rst signal to arrive from an earthquake is the P (Primary) wave followed by other P waves traveling along different paths. The P arrivals are followed by the S (Secondary) wave and by higher amplitude dispersive surface waves. The amplitude of these oscillations can increase by more than two orders of magnitude within a few minutes of the arrival of the P. The spectral components of such a time series clearly have a strong dependence on time. It would be desirable to have a joint time-frequency representation (TFR). This correspondence proposes a new transform (called the S transform) that provides a TFR with frequency-dependent resolution while, at the same time, maintaining the direct relationship, through time-averaging, with the Fourier spectrum. Several techniques of examining the time-varying nature of the spectrum have been proposed in the past; among them are the Gabor transform [7], the related short time Fourier transforms, the continuous wavelet transform [8] and the bilinear class of time-frequency distributions known as Cohen's class [4], of which the Wigner distribution [9] is a member.

II. The S Transform

There are several methods of arriving at the S transform. We consider it illuminating to derive the S transform as the "phase correction" of the continuous wavelet transform (CWT). The CWT $W(\tau, d)$ of a function $h(t)$ is de£ned by

$$W(\tau, d) = \int_{-\infty}^{\infty} h(t) w(t-\tau, d) dt \tag{1}$$

where $w(t, d)$ is a scaled replica of the fundamental mother wavelet. The dilation d determines the "width" of the wavelet $w(t, d)$ and thus controls the resolution. Along with (1), there exists an admissibility condition on the mother wavelet $w(t, d)$ [5] that $w(t, d)$ must have zero mean. The reader is referred to Rioul and Vetterli [10] and Young [11] for reviews of the literature.

The S transform of a function $h(t)$ is de£ned as a CWT with a speci£c mother wavelet multiplied by the phase factor

$$S(\tau, f) = e^{i2\pi f\tau} W(\tau, d) \tag{2}$$

where the mother wavelet is de£ned as

$$w(t, f) = \frac{|f|}{\sqrt{2\pi}} e^{-\frac{t^2 f^2}{2}} e^{-i2\pi f t}. \tag{3}$$

Note that the dilation factor d is the inverse of the frequency f.

The wavelet in (3) does not satisfy the condition of zero mean for an admissible wavelet; therefore, (2) is not strictly a CWT. Written out explicitly, the S transform is

$$S(\tau, f) = \int_{-\infty}^{\infty} h(t) \frac{|f|}{\sqrt{2\pi}} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-i2\pi f t} dt. \tag{4}$$

If the S transform is indeed a representation of the local spectrum, one would expect a simple operation of averaging the local spectra over time to give the Fourier spectrum. It is easy to show that

$$\int_{-\infty}^{\infty} S(\tau, f) d\tau = H(f) \tag{5}$$

(where $H(f)$ is the Fourier transform of $h(t)$). It follows that $h(t)$ is exactly recoverable from $S(\tau, f)$. Thus,

$$h(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} S(\tau, f) d\tau \right\} \; e^{i2\pi f t} df. \tag{6}$$

This is clearly distinct from the concepts of the CWT.

The S transform provides an extension of *instantaneous frequency* (IF) [2] to broadband signals. The one dimensional function of the variable τ and £xed parameter f_1 de£ned by $S(\tau, f_1)$ is called a voice (as with the CWT). The voice for a particular frequency f_1 can be written as

$$S(\tau, f_1) = A(\tau, f_1) e^{i\Phi(\tau, f_1)}. \tag{7}$$

Since a voice isolates a speci£c component, one may use the phase in (7) to determine the IF as de£ned by Bracewell [2].

$$IF(\tau, f_1) = \frac{1}{2\pi} \frac{\partial}{\partial \tau} \{2\pi f_1 \tau + \Phi(\tau, f_1)\} \tag{8}$$

Thus the absolutely referenced phase information leads to a generalization of the IF of Bracewell to broadband signals. The validity of (8) can easily be seen for the simple case of $h(t) = cos(2\pi w t)$, where the function $\Phi(\tau, f) = 2\pi(w - f)\tau$.

The linear property of the S transform insures that for the case of additive noise (where one can model the data as *data(t) = signal(t) + noise(t)*), the S transform gives

$$S\{data\} = S\{signal\} + S\{noise\}.$$

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R. G. Stockwell and R. P. Lowe are with the Canadian Network for Space Research and Department of Physics, The University of Western Ontario, London, Ontario, Canada N6A 5B7 (Email: stockwell@colorado-research.com)

L. Mansinha is with the Department of Earth Sciences, The University of Western Ontario, London, Ontario, Canada N6A 5B7

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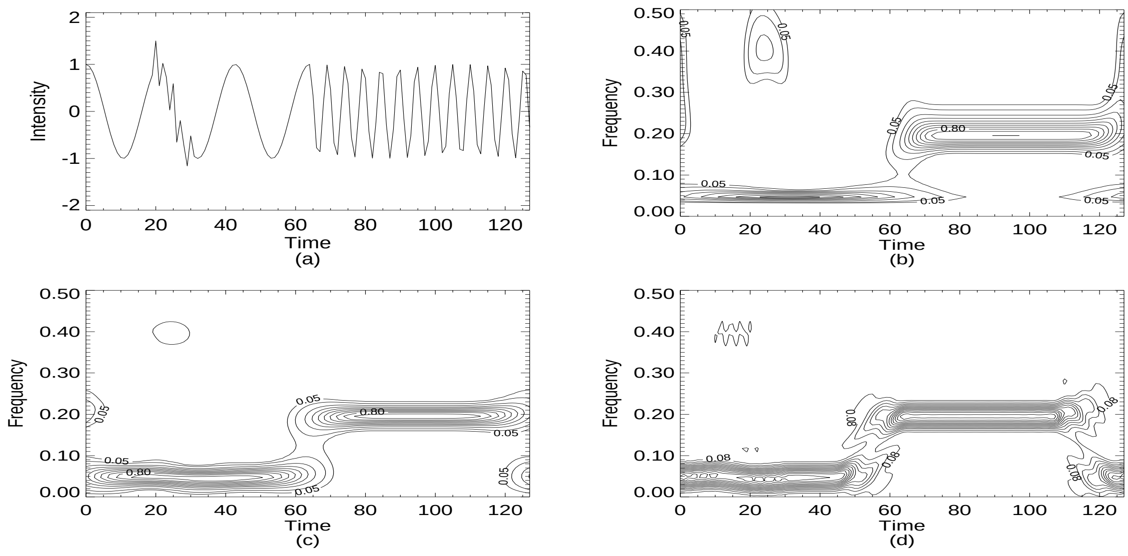

Fig. 1. (a) A synthetic time series consisting of a low frequency signal for the £rst half, a middle frequency signal for the second half, and a high frequency burst at t=20. The function is $h[0:63] = cos(2\pi t * 6.0/128.0)$, $h[63:127] = cos(2\pi t * 25.0/128.0)$, $h[20:30] = h[20:30] + 0.5 * cos(2\pi t * 52.0/128.0)$. (b) The amplitude of the S transform of the time series of Fig 1(a). Note the good frequency resolution for the $f = 6/128$ signal, as well as the good time resolution of the $f = 52/128$ signal. (c) The Short Time Fourier transform (STFT) of the time series in Fig 1(a) using a £xed gaussian window of standard deviation = 8 units. The high frequency burst is smoothed out because of the poor time resolution of the STFT. (d) Same as Fig 1(c) except that the window is a boxcar of length = 20 units .

This is an advantage over the bilinear class of TFRs (see "propagation of noise" in [4]), where one £nds

$$TFR\{data\} = TFR\{signal\} + TFR\{noise\}$$
$$+2 * TFR\{signal\} * TFR\{noise\}.$$

The presence of the cross terms makes it dif£cult to reliably estimate the signal. As with standard spectral analysis, signi£cance test must be performed on the noisy local spectra to establish con£dence intervals.

The S transform can be written as operations on the Fourier spectrum $H(f)$ of $h(t)$

$$S(\tau, f) = \int_{-\infty}^{\infty} H(\alpha + f)\, e^{-\frac{2\pi^2\alpha^2}{f^2}} e^{i2\pi\alpha\tau} d\alpha \quad f \neq 0. \qquad (9)$$

The discrete analog of (9) is used to compute the discrete S transform by taking advantage of the ef£ciency of the Fast Fourier transform (FFT) and the convolution theorem. By not translating the cosinusoid basis functions, the S transform localizes the real and imaginary components of the spectrum independently, localizing the phase spectrum as well as the amplitude spectrum.

III. The Discrete S Transform

Let $h[kT]$, $k = 0,1,...,N-1$ denote a discrete time series, corresponding to $h(t)$, with a time sampling interval of T. The discrete Fourier transform is given by [3]

$$H[\frac{n}{NT}] = \frac{1}{N} \sum_{k=0}^{N-1} h[kT] e^{-\frac{i2\pi nk}{N}} \qquad (10)$$

where $n = 0,1,...,N-1$. In the discrete case, the S transform is the projection of the vector de£ned by the time series $h[kT]$ onto a spanning set of vectors. The spanning vectors are not orthogonal, and the elements of the S transform are not independent. Each basis vector (of the Fourier transform) is divided into N localized vectors by an element-by-element product with the N shifted Gaussians, such that the sum of these N localized vectors is the original basis vector.

Using (9), the S transform of a discrete time series $h[kT]$ is given by (letting $f \rightarrow n/NT$ and $\tau \rightarrow jT$)

$$S[jT, \frac{n}{NT}] = \sum_{m=0}^{N-1} H[\frac{m+n}{NT}] e^{-\frac{2\pi^2 m^2}{n^2}} e^{\frac{i2\pi mj}{N}} \quad n \neq 0 \qquad (11)$$

and for the $n = 0$ voice, it is equal to the constant de£ned as

$$S[jT, 0] = \frac{1}{N} \sum_{m=0}^{N-1} h(\frac{m}{NT}) \qquad (12)$$

where j, m and $n = 0, 1, ..., N-1$. Equation (12) puts the constant average of the time series into the zero frequency voice, thus assuring the inverse is exact. The discrete S transform suffers the familiar problems from sampling and £nite length, giving rise to implicit periodicity in the time and frequency domains. The discrete inverse of the S transform is:

$$h[kT] = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{j=0}^{N-1} S[jT, \frac{n}{NT}] \right\} e^{\frac{i2\pi nk}{N}}. \qquad (13)$$

IV. Self-Aliasing

The Fourier spectrum of the Gaussian window at a speci£c n is called a *voice Gaussian*. Using (11), any voice of the S transform, may be computed as the product of the Fourier spectrum $H[m/NT]$ of the time series and the voice Gaussian. For example, in (11), the voice Gaussian is

$$G(m, n) = e^{-\frac{m^2}{2(\frac{n}{2\pi})^2}}. \qquad (14)$$

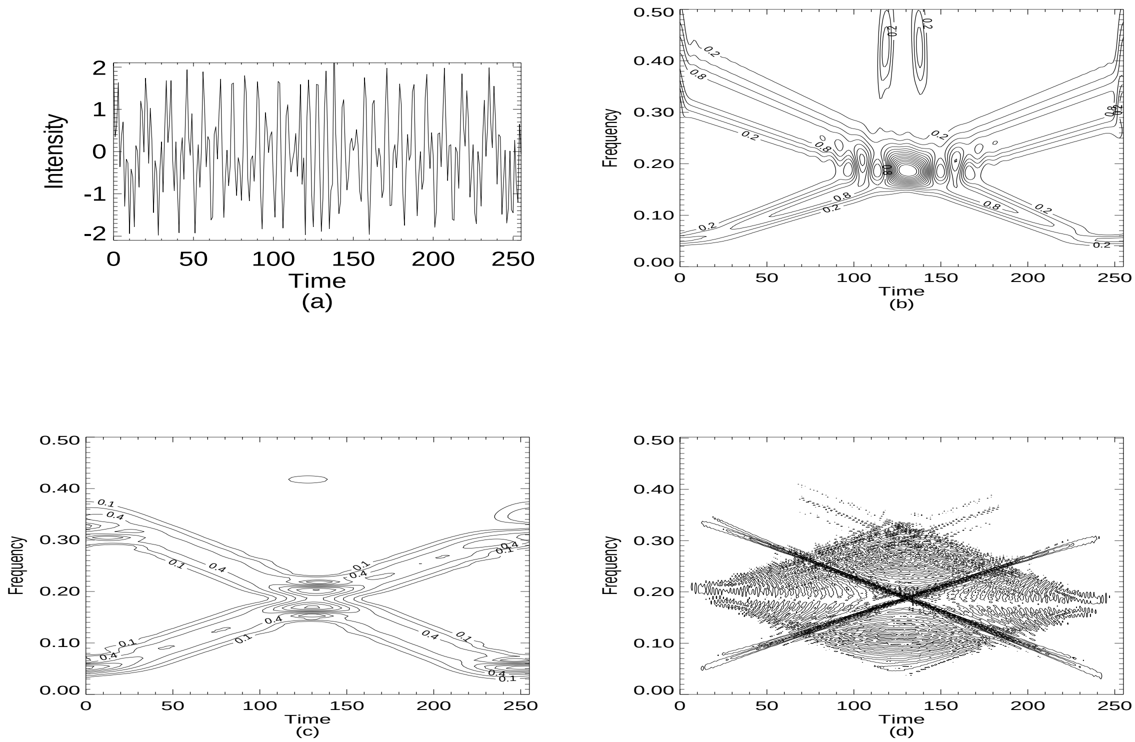

Fig. 2. (a) A synthetic time series consisting of two cross chirps and two high frequency bursts. The time series is: $h[0:255] = cos(2\pi(10 + t/7) * t/256) + cos(2\pi(256/2.8 - t/6.0) * t/256)$ $h[114:122] = h[114:122] + cos(2\pi t * 0.42)$ $h[134:142] = h[134:142] + cos(2\pi t * 0.42)$ (b) The amplitude of the S transform of the time series in Fig 2(a). The two chirps, as well as both high frequency bursts, are seen. (c) The amplitude of the STFT (with a Gaussian window) of the time series in Fig 2(a). Both chirps are detected, but the two high frequency bursts are not detected. (d) The amplitude of the Wigner distribution of the time series in Fig 2(a). Both chirps are detected with very good resolution in both time and frequency. The two high frequency bursts are not detected.

Due to sampling in the time domain, the discrete spectrum is periodic. Thus, in (11) one can see that as n/NT approaches the Nyquist frequency $1/2T$, the voice Gaussian overlaps into the negative frequencies of $H[m/NT]$. We call this *self-aliasing.* This self-aliasing occurs even when the sampling rate satis£es the Nyquist criterion. The voice Gaussian becomes quite wide at the high frequencies, and self aliasing introduces errors. In order to minimize the effect of self-aliasing, we de£ne a special Nyquist frequency for the S transform s_N. When the voice Gaussian is centred over s_N, the 2σ point is at the Fourier Nyquist frequency f_N. Thus

$$s_n + 2\frac{s_n}{2\pi} = f_n \tag{15}$$

The minimum sample rate T_s is given by

$$T_s = \frac{\pi + 1}{2\pi f_N} \tag{16}$$

Compare to the Fourier Nyquist sample rate given by $T_f = 1/(2f_N)$. Of course, when dealing with a real-valued time series, one can use the analytic signal of the time series for localization [1]. This has the result of setting all negative frequencies to zero, and thus there is no self-aliasing.

V. Discussion

Figs. 1 and 2 demonstrate the class of time series for which the S transform would be useful; they highlight the advantages of such an approach as compared with other techniques.

In Fig. 1, a synthetic time series consisting of high and low frequencies and a high frequency burst is used to compare the performance of the S transform with the STFT. The frequency dependent resolution of the S transform allows the detection of the high frequency burst, and shows good frequency resolution on the long period signal. The constant window width of the STFT shows up in the poorer de£nition of all three components. The long period signal in the STFT is self-aliased, causing a variation in its amplitude. With the STFT there is a tradeoff between detecting short lived high frequency signals and low frequency signals.

In Fig. 2, two high-frequency bursts are added to crossed chirps. All four components are detected by the S transform. The chirps are seen with the STFT, but the two high frequency components are not detected. The STFT does not have suf£cient time resolution to resolve the two signals. In fact, the interference of the two bursts causes it to appear as a single signal at a lower frequency. The Wigner distribution has very good time and frequency resolution, as is evident on the crossed chirps. The cost of such resolution is the presence of cross terms which hinder interpretation of such a plot. Indeed, the interference of cross terms causes the two bursts to be completely misrepresented.

VI. Conclusions

The power and limitations of the S transform are apparent from the above examples. The inverse frequency dependence of the localizing Gaussian window is an improvement over the £xed width window used in the STFT. The phase of the S transform referenced to the time origin provides useful and supplementary information about spectra which is not available from locally referenced phase information in CWT. However, being related to the Fourier transform, the S transform suffers from some of the same drawbacks that af¤ict all discretely sampled transforms. We believe that further developments in S transform will £nd applications in a broad range of disciplines.

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